

Lecture 7

The Eigen- & Singular Value Decompositions

Nonlinear matrix algorithms

**CS328 - Numerical Methods for
Visual Computing and Machine Learning**

Prof. Wenzel Jakob

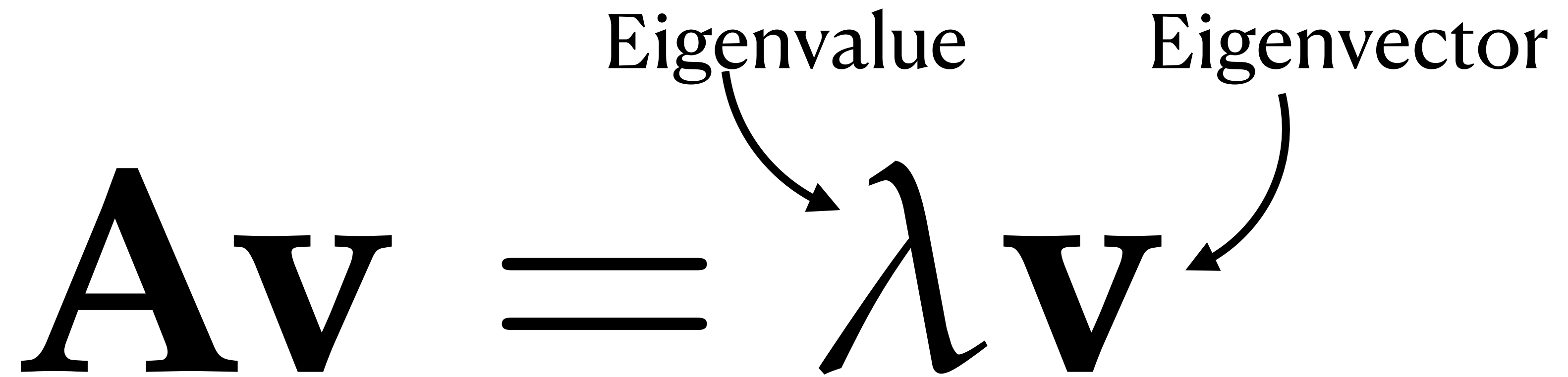
MATH-111 recap

$$A\mathbf{v} = \lambda\mathbf{v}$$

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Eigenvalue Eigenvector

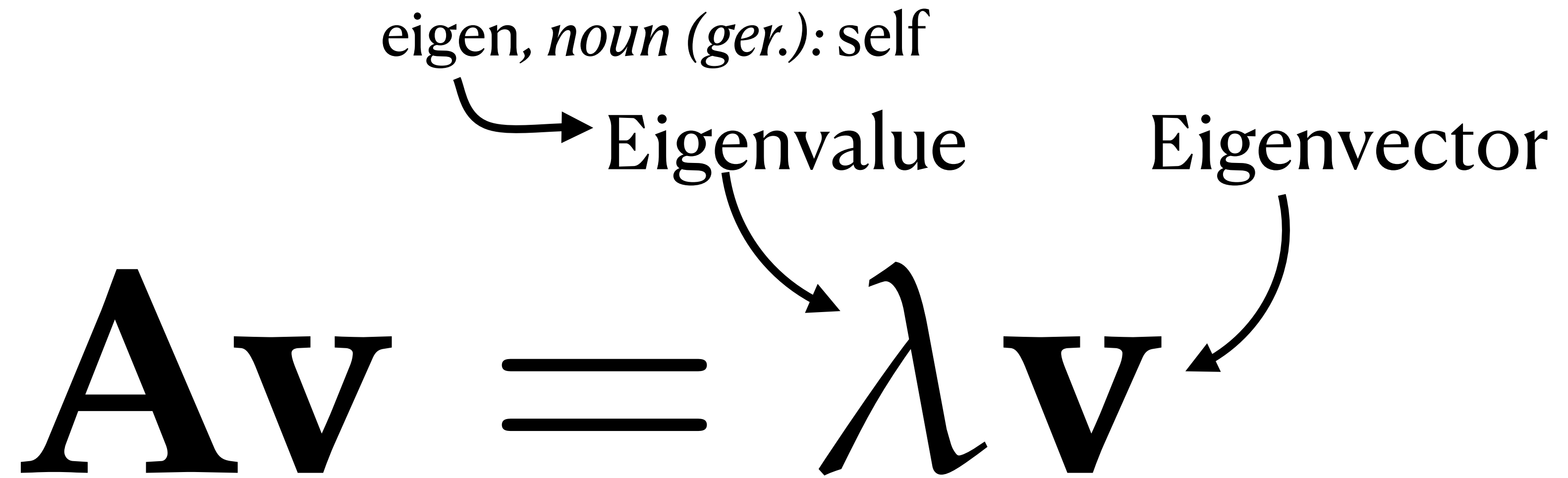
The diagram shows the eigenvalue equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. The term λ is labeled "Eigenvalue" with a curved arrow pointing to it. The term \mathbf{v} is labeled "Eigenvector" with a curved arrow pointing to it.

MATH-111 recap

eigen, *noun* (ger.): self

Eigenvalue

Eigenvector

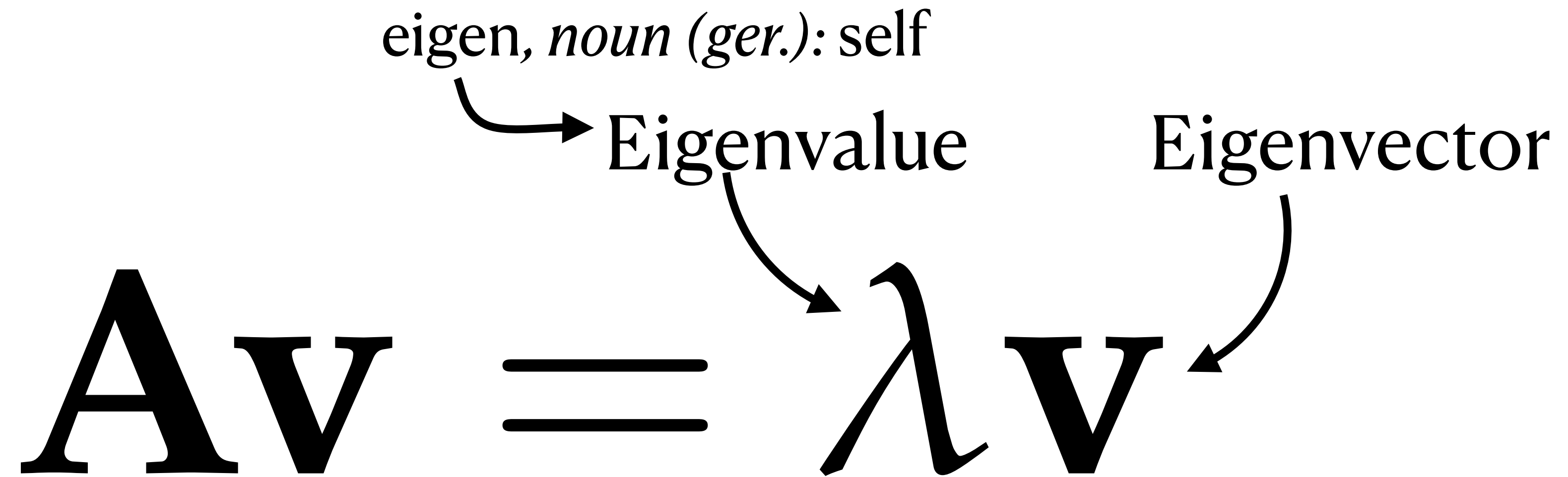
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where

$$\mathbf{A} \in \mathbb{R}^{n \times n}, \lambda \in \mathbb{R}$$

$$\mathbf{v} \in \mathbb{R}^n, \|\mathbf{v}\| = 1$$

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$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

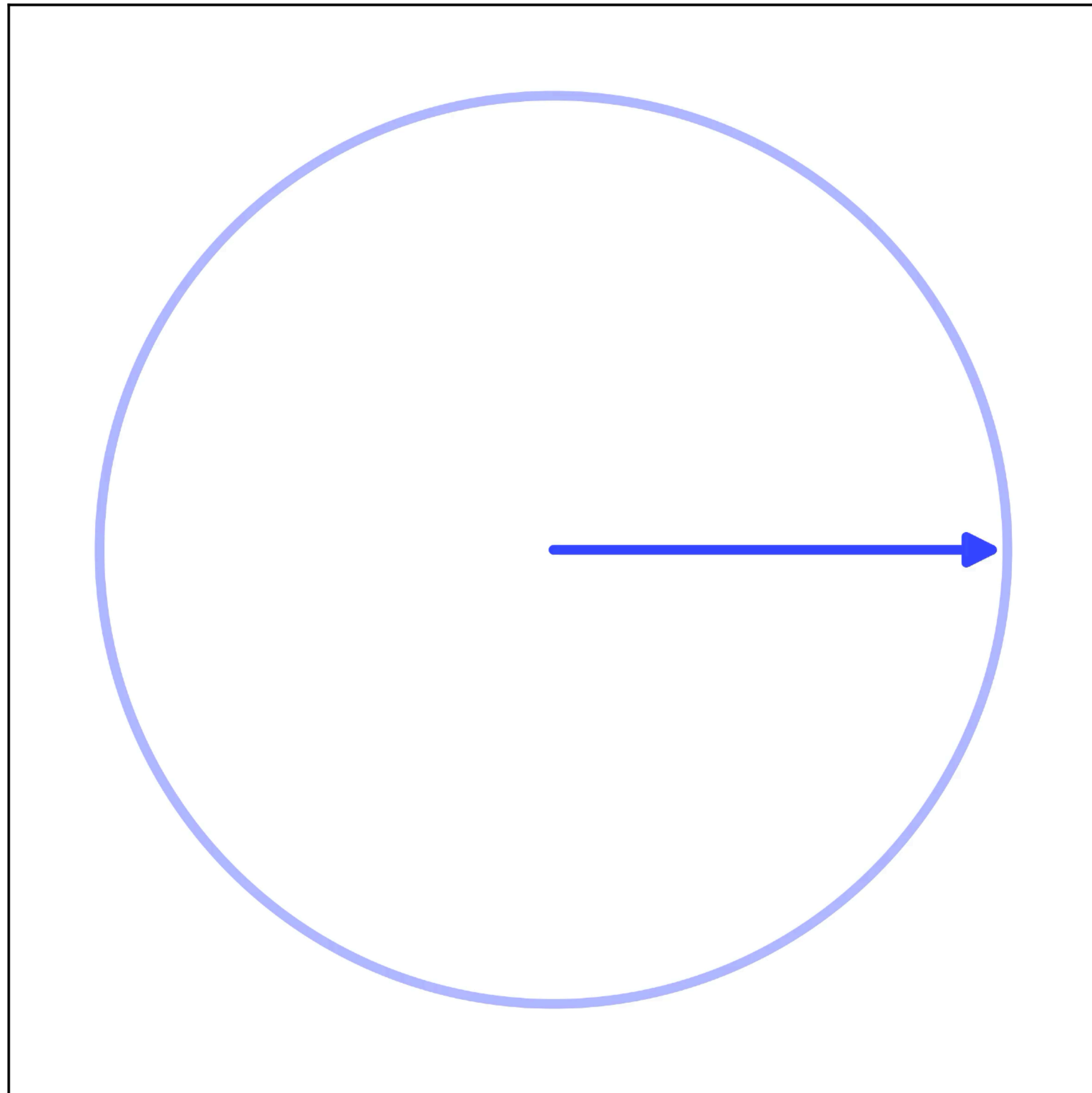
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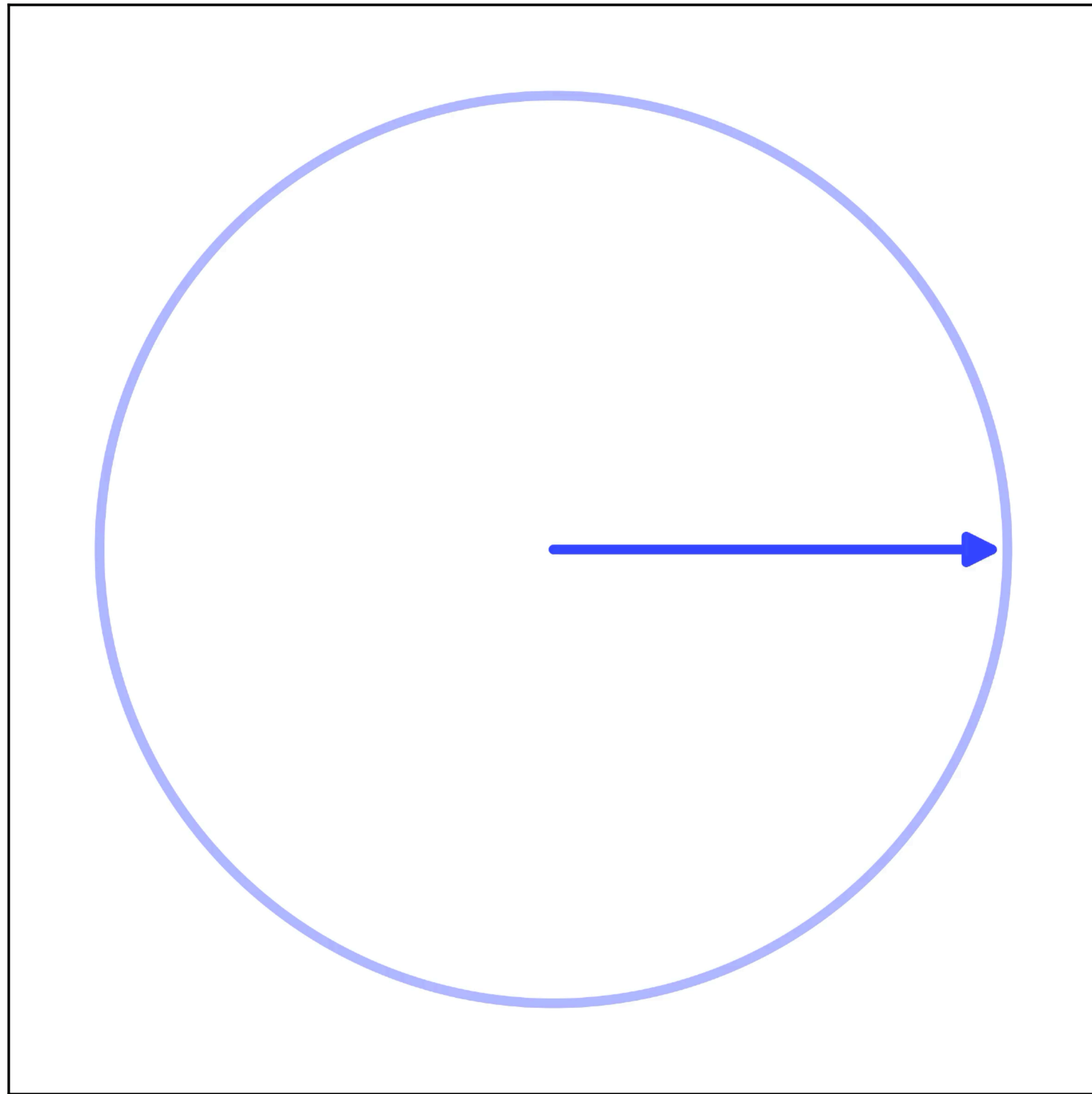
$$\mathbf{v}_i \in \mathbb{R}^n, \|\mathbf{v}_i\| = 1$$

$$(i = 1, \dots, n)$$

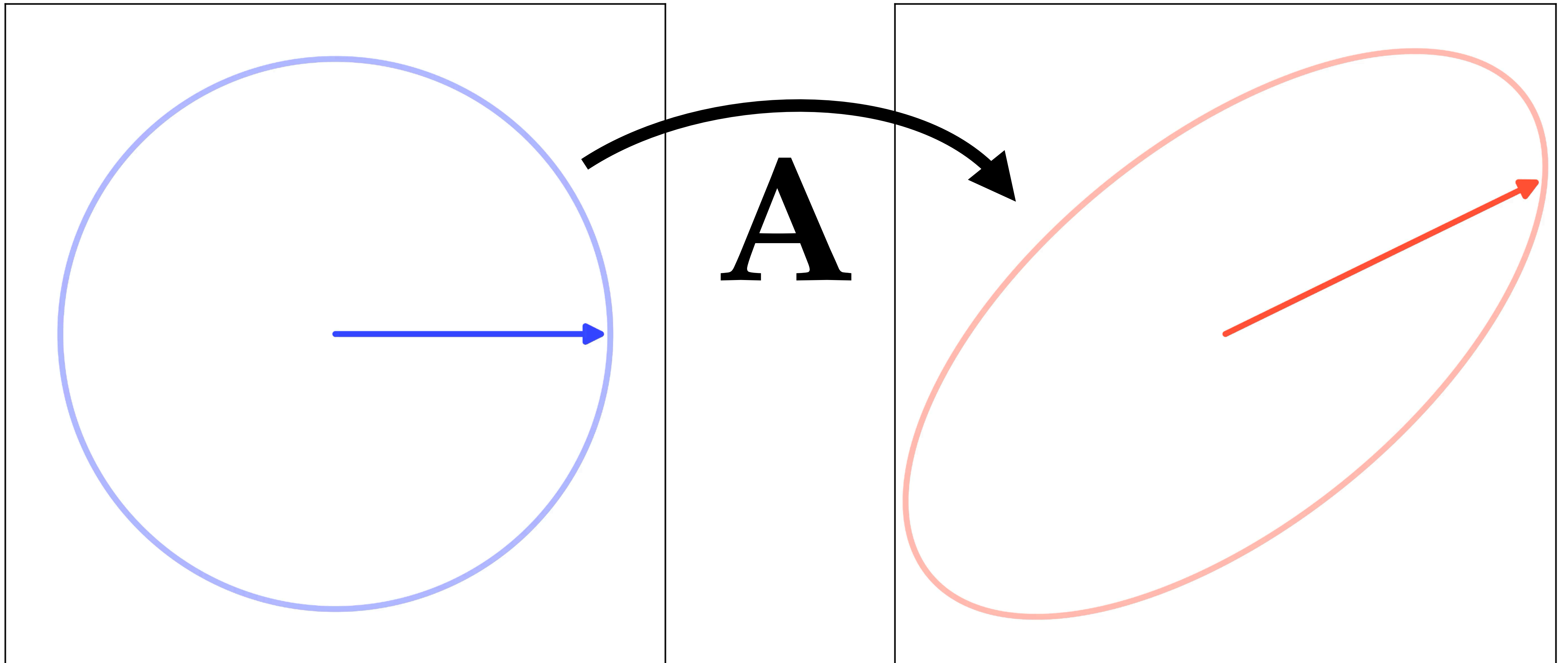
Eigendecomposition: the geometry



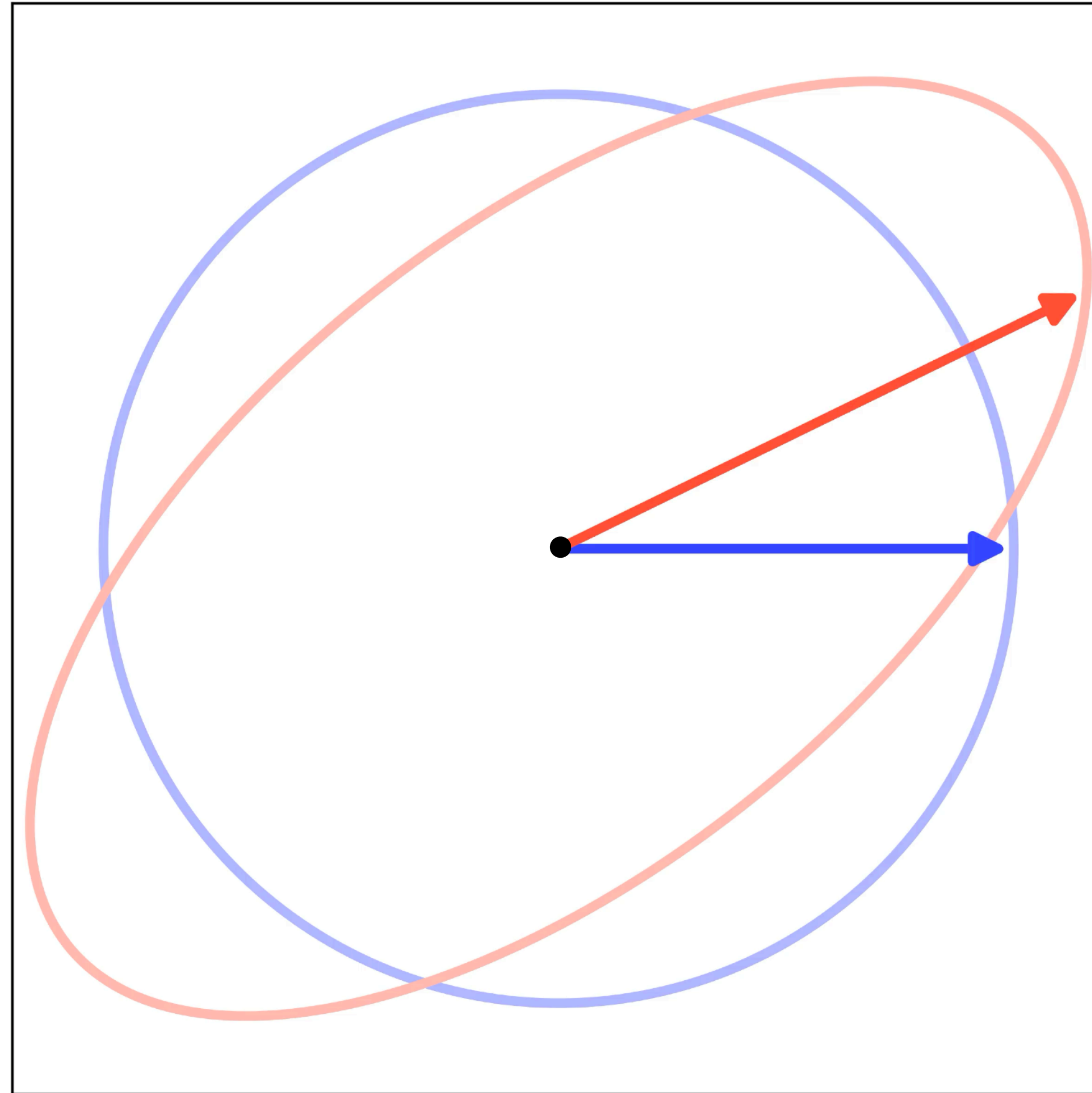
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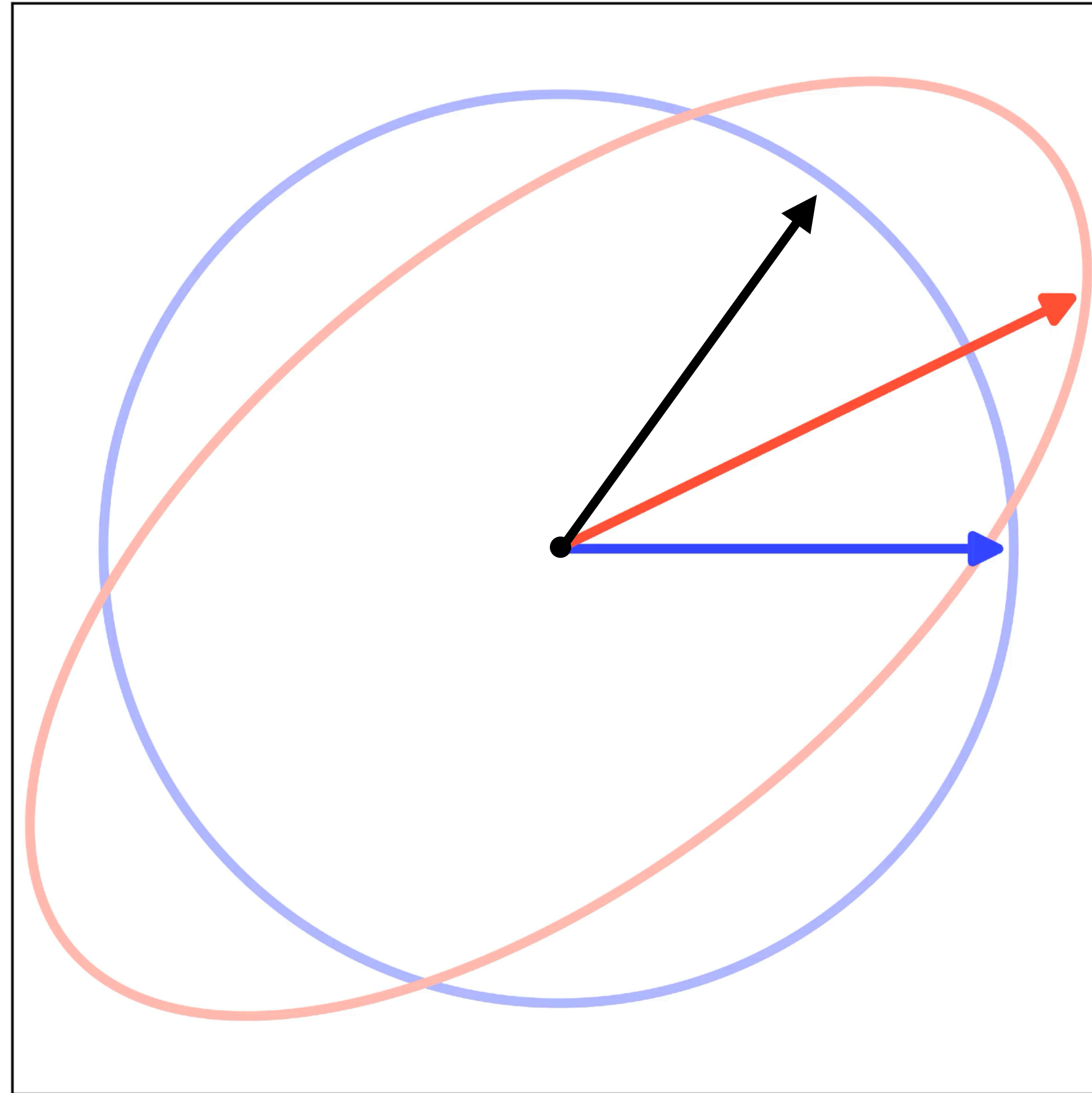
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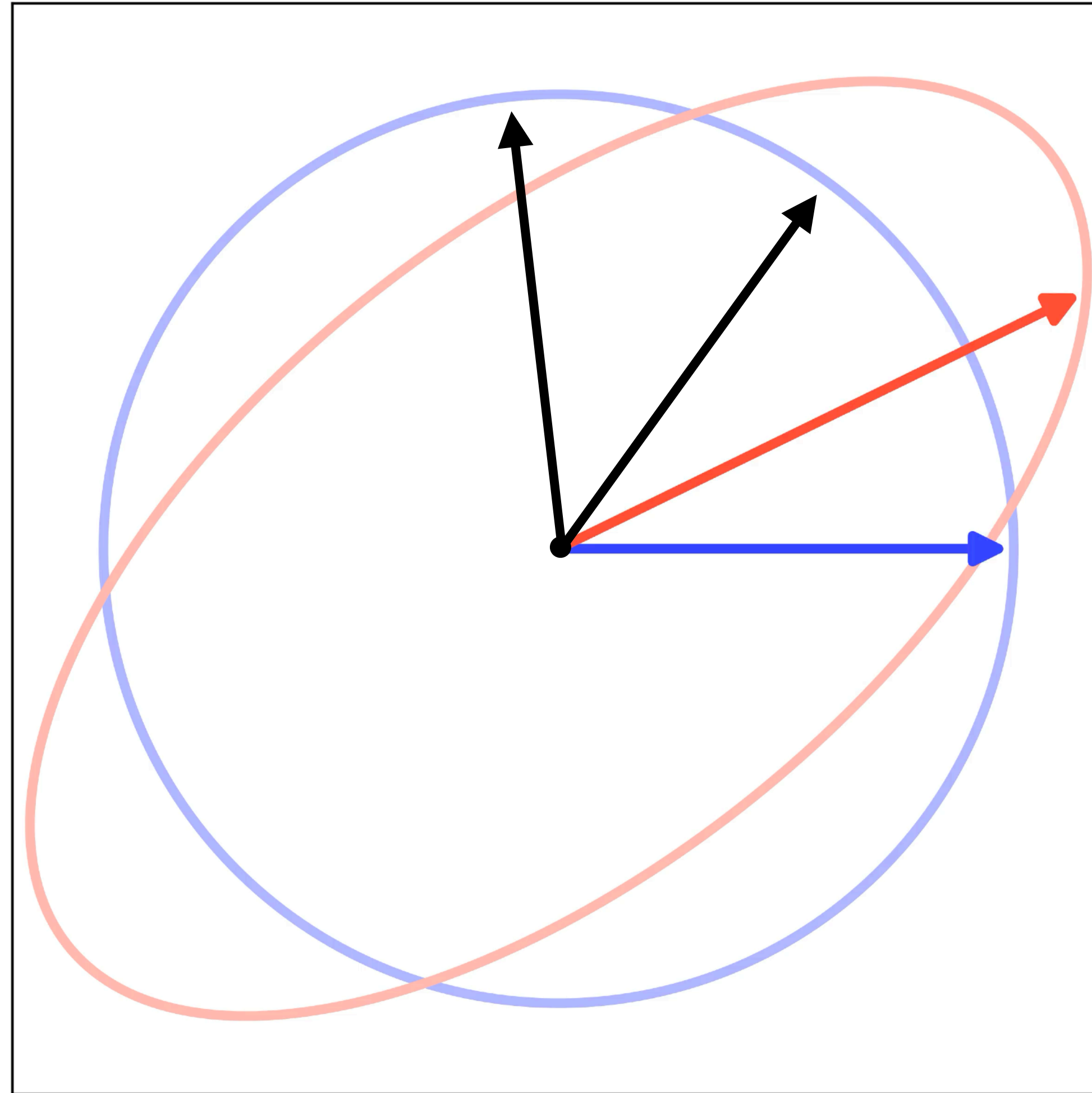
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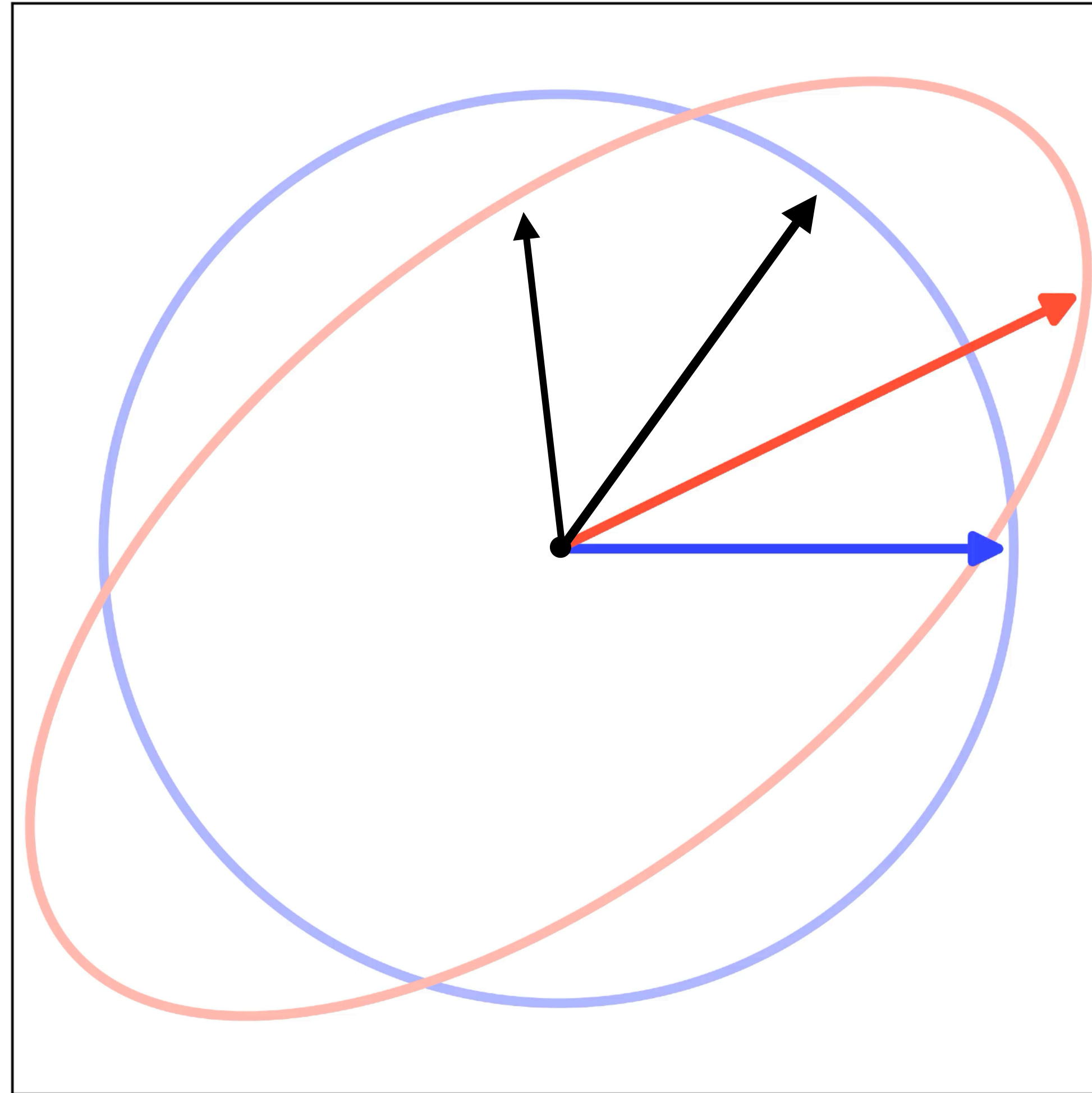
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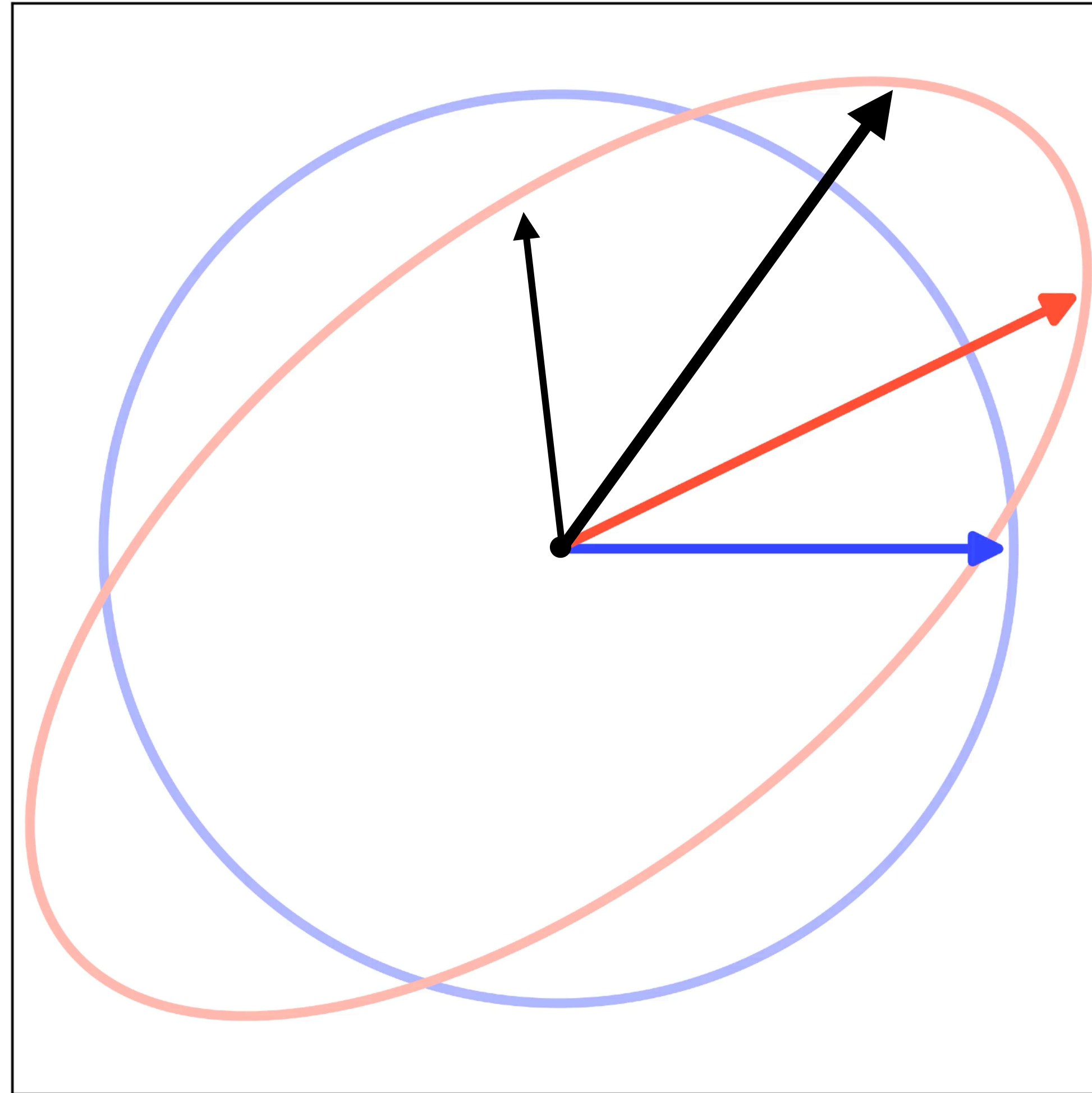
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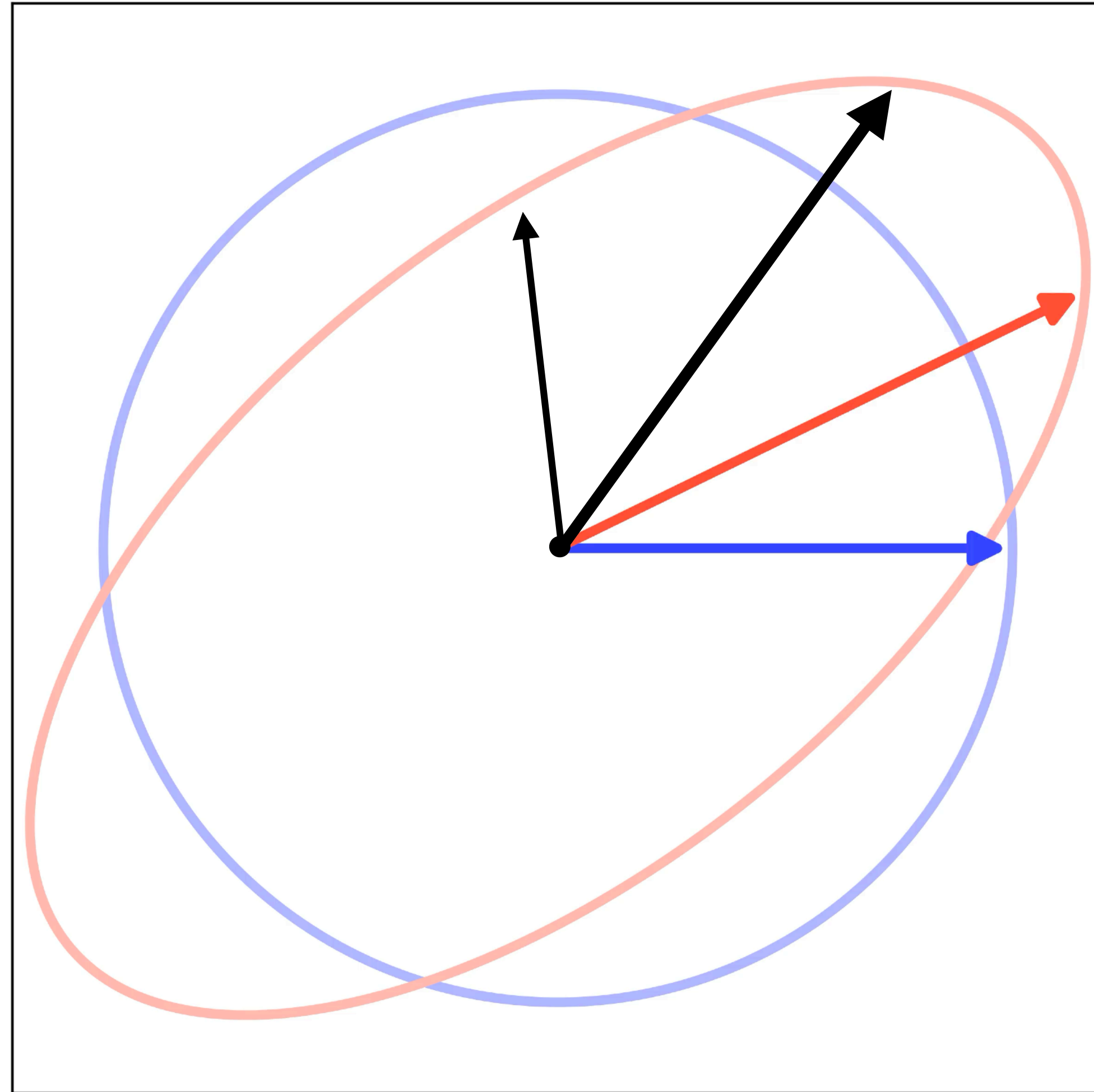
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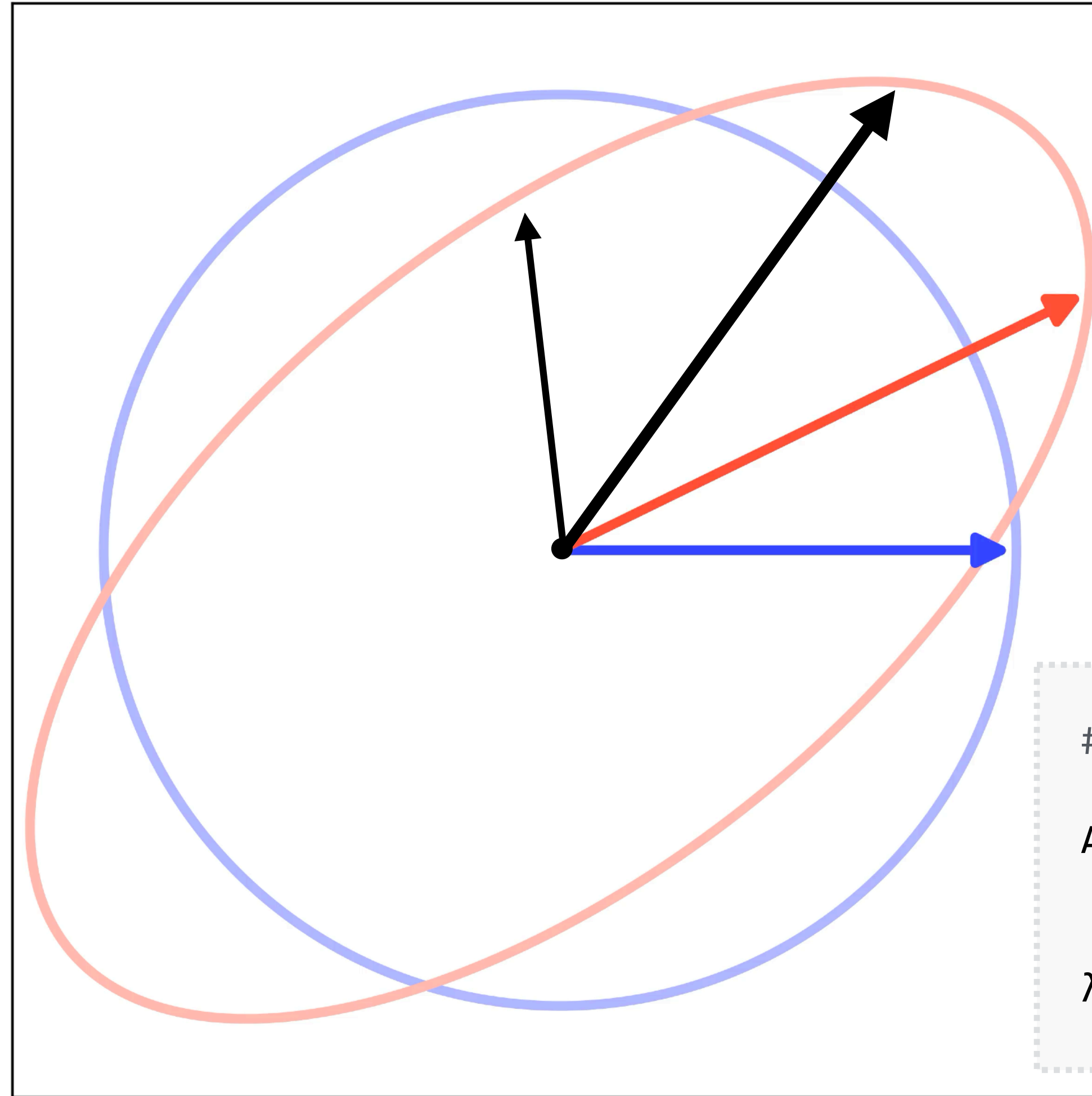
Eigendecomposition: the geometry



$$\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$

$$\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$

Eigendecomposition: the geometry



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```
# Randomly generated matrix
```

```
A = array([[ 1.16043581,  0.0566787 ],  
          [ 0.56722123,  0.85777919]])
```

```
 $\lambda$ , V = scipy.linalg.eig(A)
```

Eigenvectors (.. of non-singular matrices)

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 - .. which are not generally orthogonal (guaranteed only if \mathbf{A} is symmetric).
- Therefore, a matrix with those \mathbf{v}_i as its columns

$$\mathbf{V} = \left(\begin{array}{c|ccc|c} & & & & \\ & & & & \\ \mathbf{v}_1 & & \cdots & & \mathbf{v}_n \\ & & & & \\ & & & & \end{array} \right)$$

has rank n (i.e., *full rank*) and so is non-singular

What happens when we multiply A and V ?

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In other words:

$$\mathbf{AV} = \mathbf{V}\Lambda$$

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The Eigendecomposition

$$\mathbf{A} = \mathbf{V} \Lambda \mathbf{V}^{-1}$$

Geometric summary

The eigendecomposition expresses a linear transformation A as product of 3 transforms

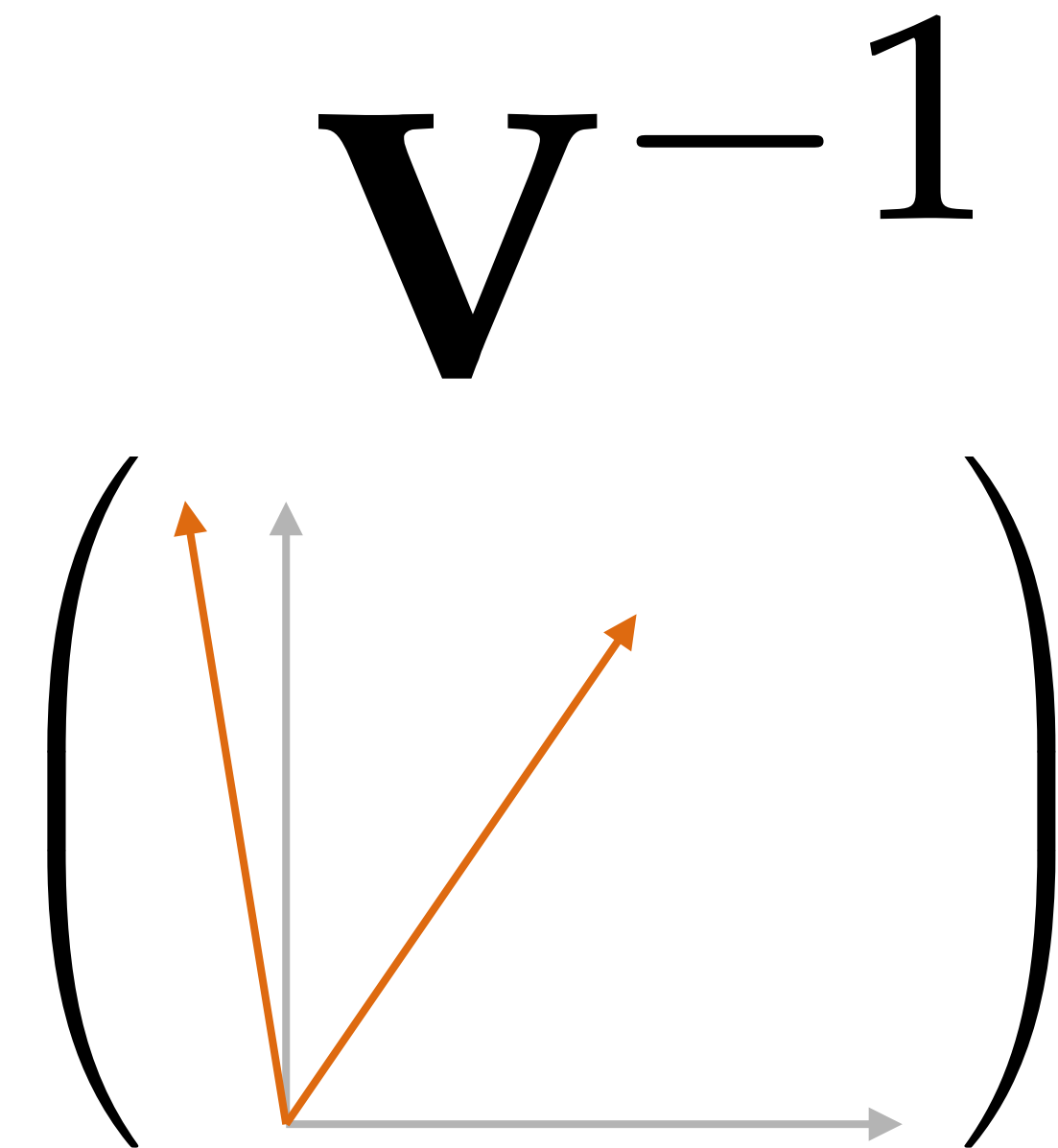
$$A =$$

$$\left(\begin{array}{c} \mathbf{V}^{-1} \\ \uparrow \\ \left(\begin{array}{c} \uparrow \\ \rightarrow \end{array} \right) \end{array} \right)$$

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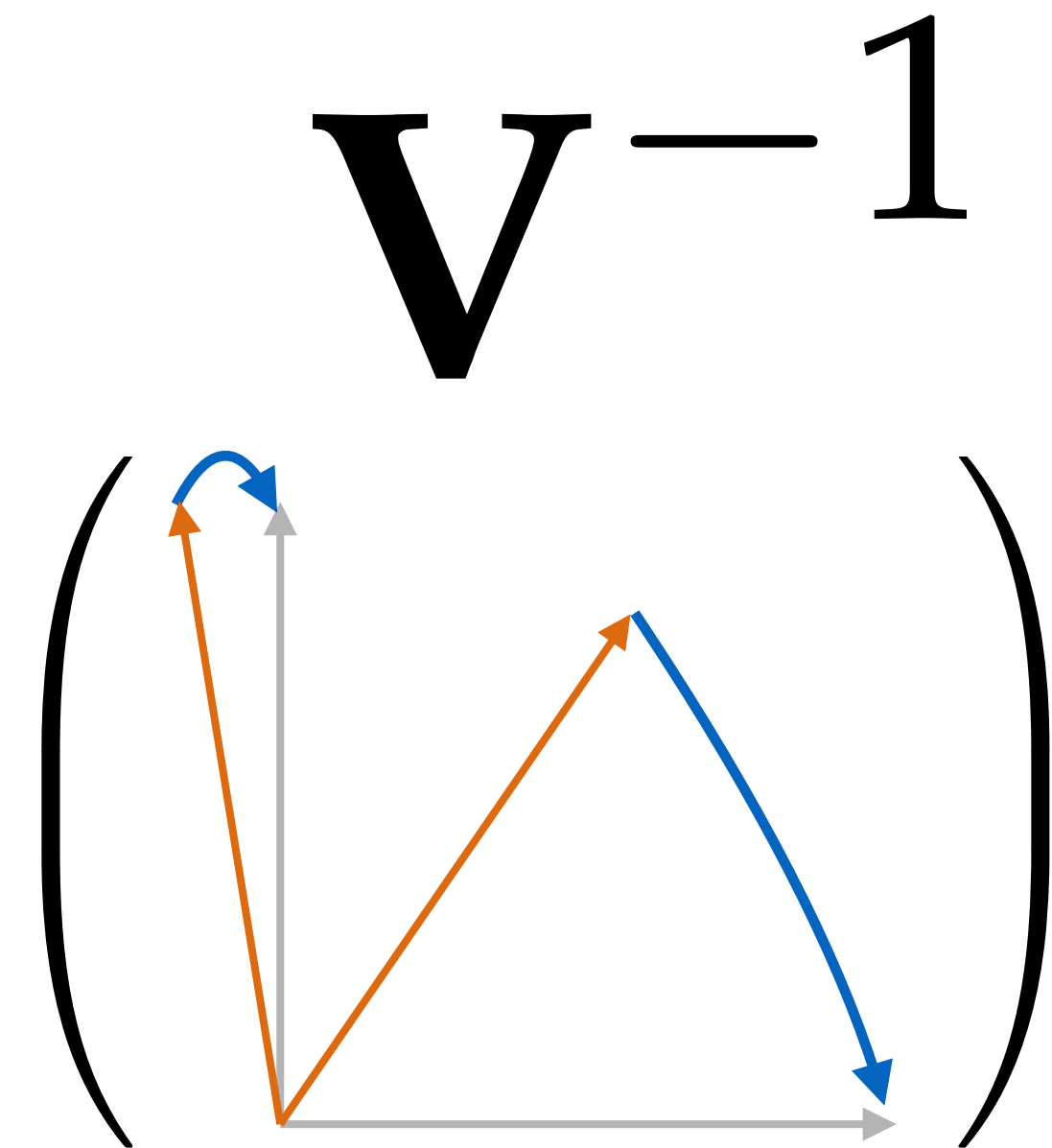


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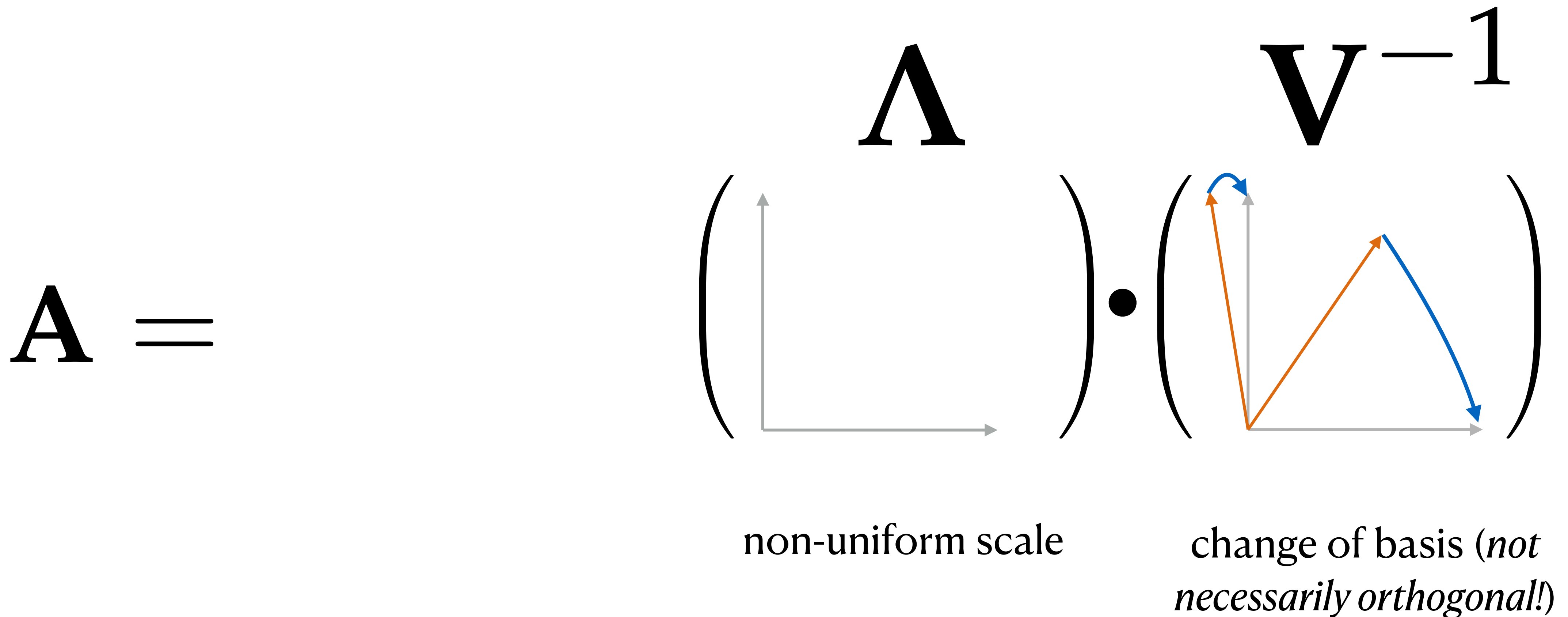
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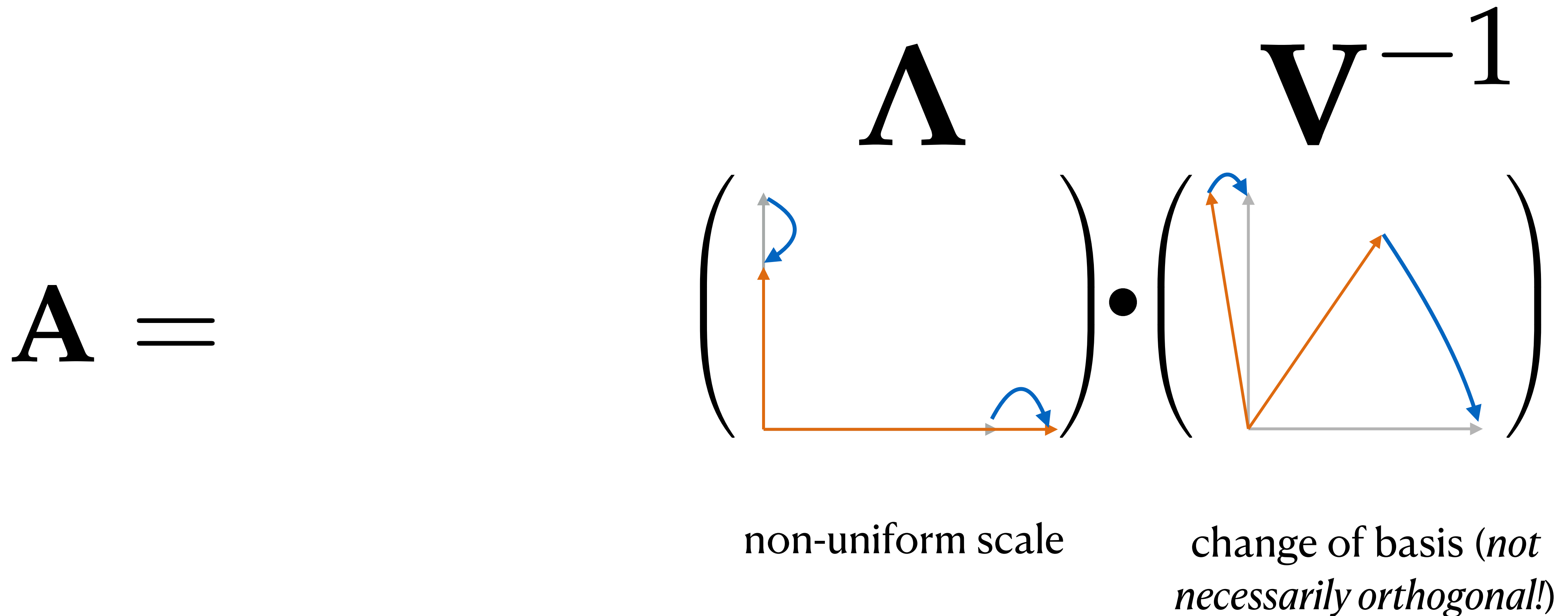
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$$\mathbf{A} = \mathbf{\Lambda} \mathbf{V}^{-1}$$

The diagram illustrates the decomposition of a linear transformation A into two components. On the left, a coordinate system is shown with a vertical axis and a horizontal axis. The vertical axis is represented by a grey arrow pointing up, and the horizontal axis by a grey arrow pointing right. A blue arrow points up along the vertical axis, and a red arrow points right along the horizontal axis, representing a non-uniform scale. Below this diagram is the text "non-uniform scale". On the right, a coordinate system is shown with a vertical axis and a horizontal axis. The vertical axis is represented by a grey arrow pointing up, and the horizontal axis by a grey arrow pointing right. A blue arrow points up and to the left, and a red arrow points up and to the right, representing a change of basis. Below this diagram is the text "change of basis (not necessarily orthogonal!)". A black dot is placed between the two diagrams, indicating their product.

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$$\mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$$

(inverse)
change of basis

non-uniform scale

change of basis (*not necessarily orthogonal!*)

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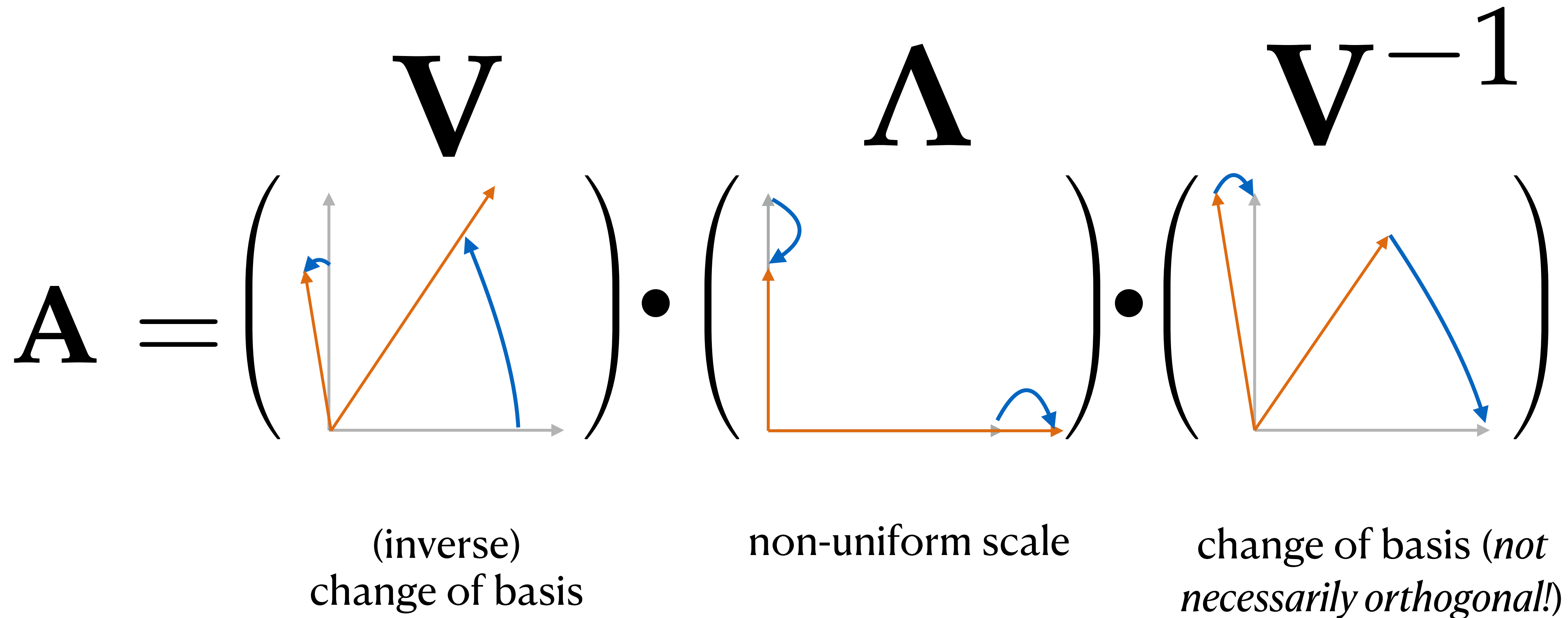
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The diagram illustrates the eigendecomposition of a linear transformation A as a product of three geometric transformations:

- (inverse) change of basis:** The first transformation, labeled \mathbf{V} , shows a 2D coordinate system with a gray horizontal axis and a gray vertical axis. A new basis is defined by two orange vectors originating from the origin. One orange vector is in the first quadrant, and the other is in the second quadrant.
- non-uniform scale:** The second transformation, labeled $\mathbf{\Lambda}$, shows the same coordinate system. The orange vectors from the previous step are scaled. The horizontal vector is scaled significantly more than the vertical vector. Blue curved arrows indicate the scaling of each vector.
- change of basis (not necessarily orthogonal!):** The third transformation, labeled \mathbf{V}^{-1} , shows the coordinate system after the scaling. The orange vectors are now rotated relative to the gray axes. Blue curved arrows indicate the rotation of each vector.

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The eigendecomposition expresses a linear transformation A as product of 3 transforms



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 - Important processing step to remove irrelevant data.

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Will see examples of all of these.
(Some in the context of SVD)

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MATH-111: Use the Characteristic Polynomial!

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0.$$

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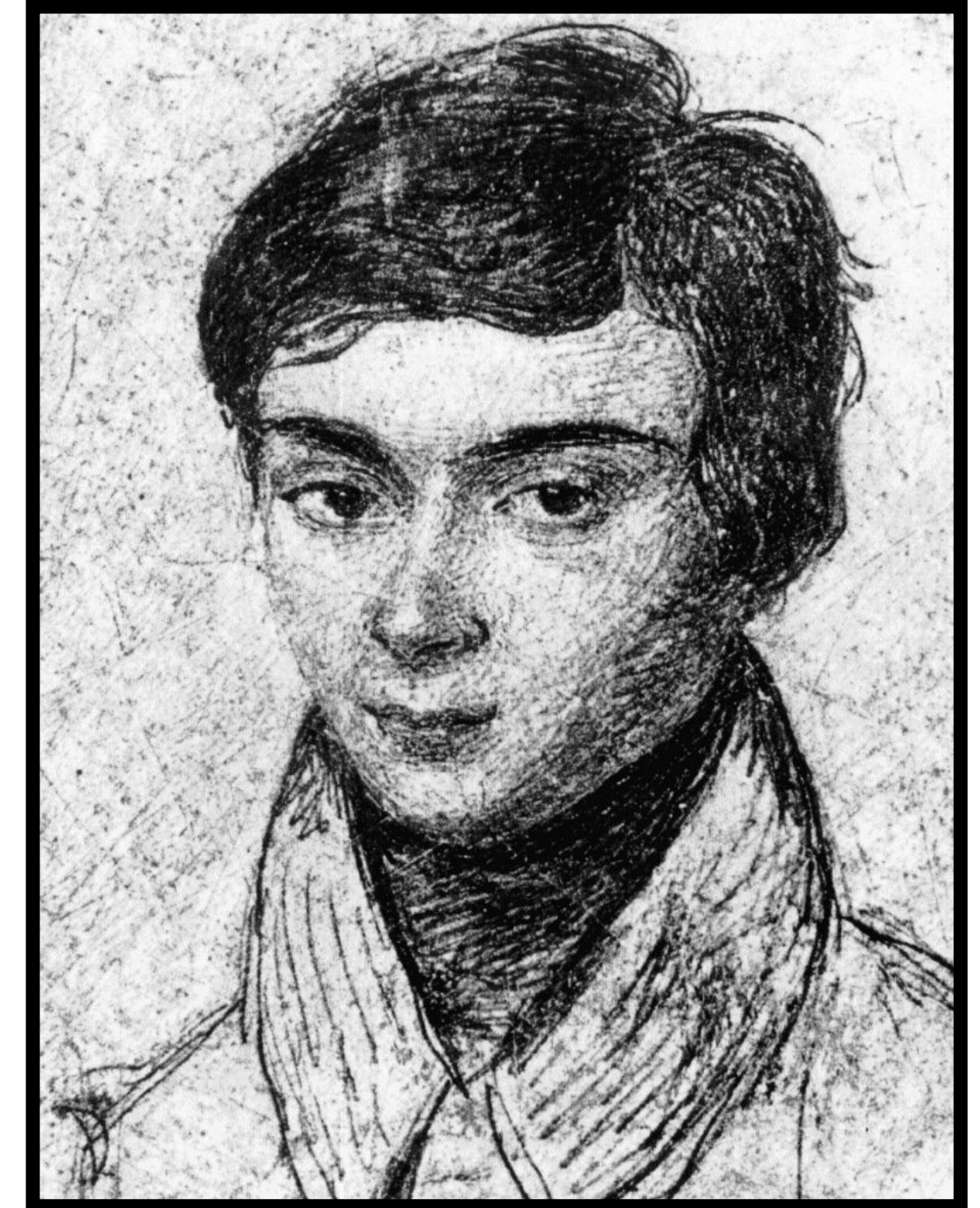
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- *Galois theory*: most polynomials of degree 5 or higher cannot be solved using *algebraic methods*.



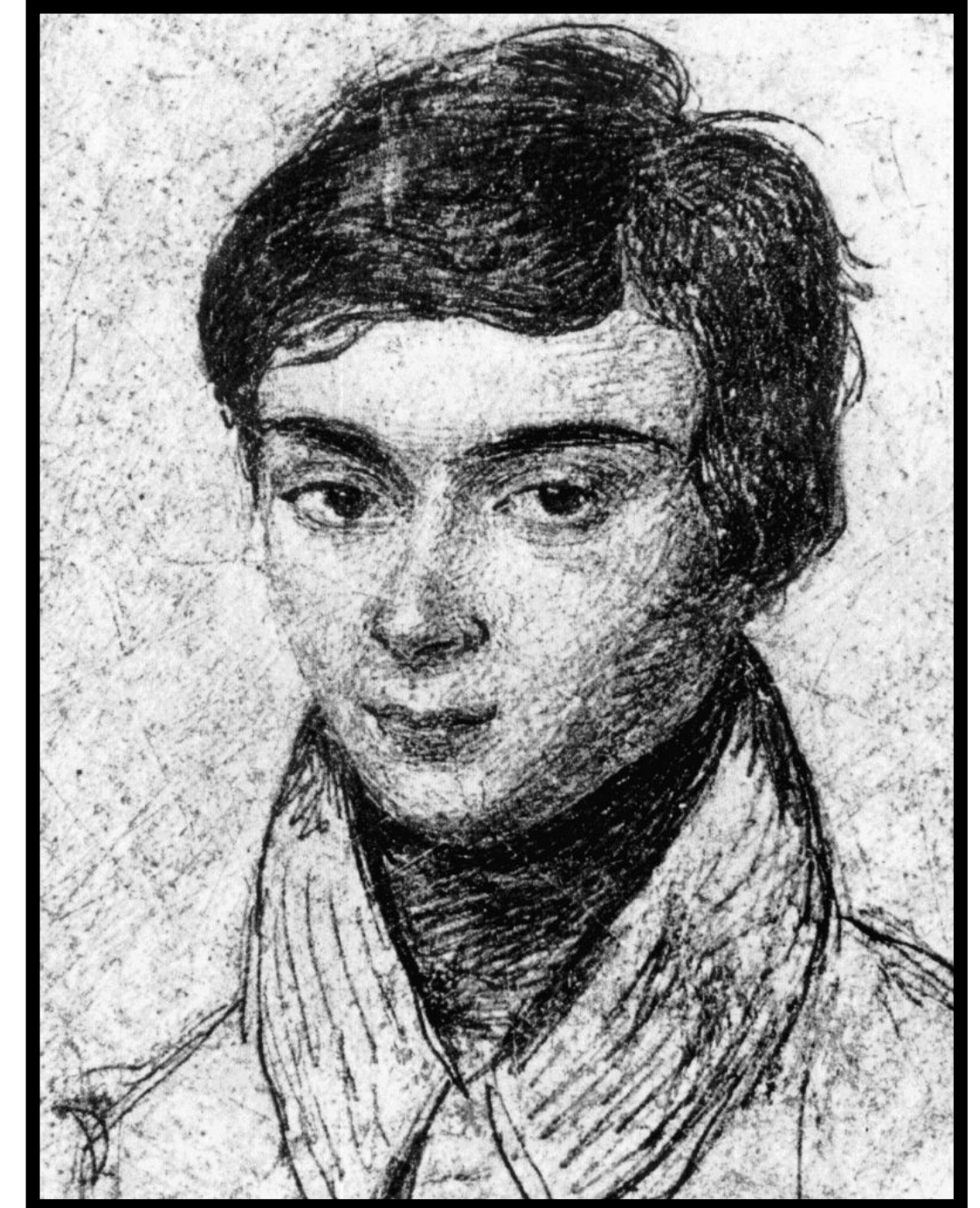
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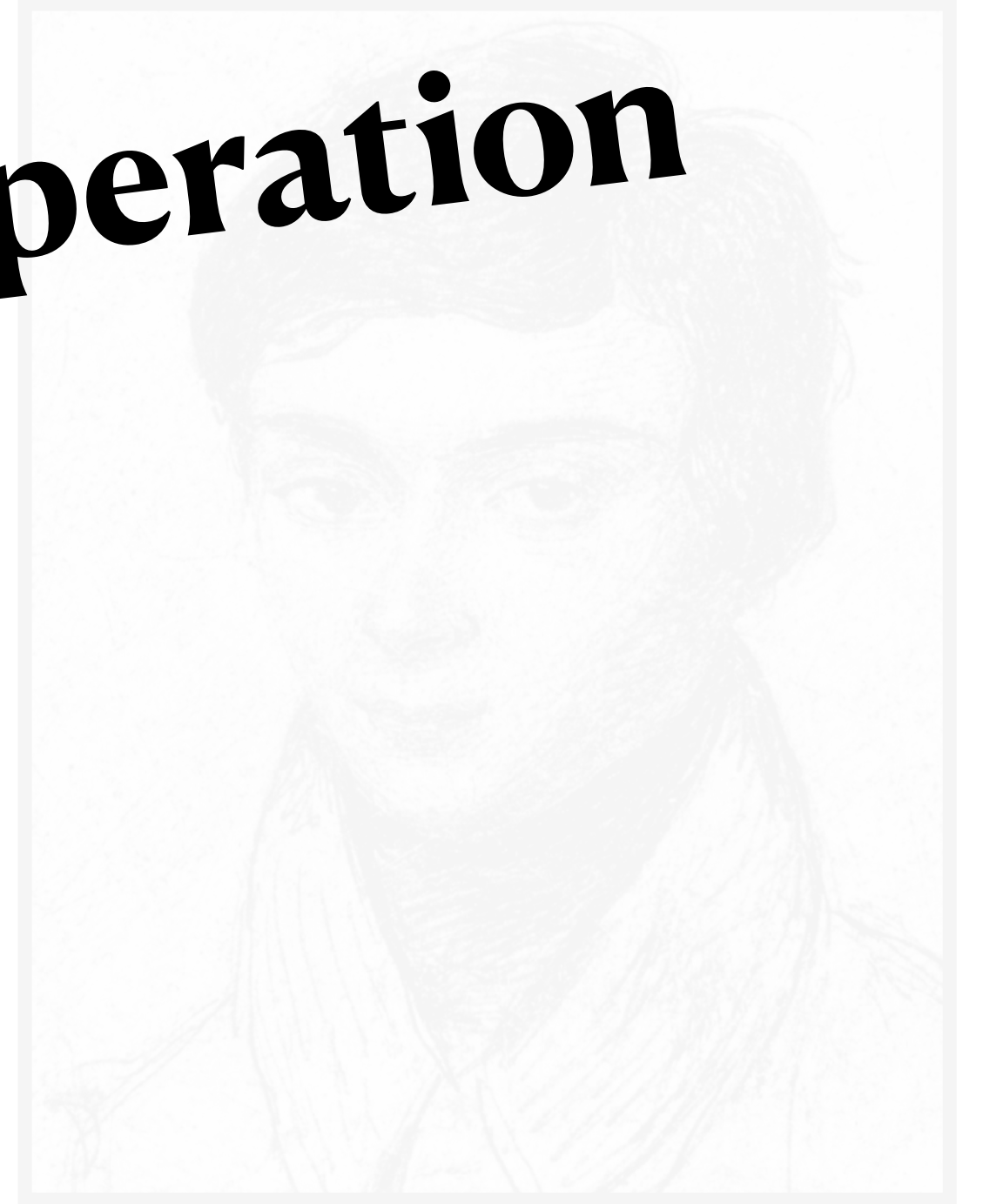
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MATH-111: Use the Characteristic Polynomial!

$$\det(\lambda I - A) = 0$$

Eigendecomposition is a nonlinear operation

- As the name indicates, this is a polynomial.
- Degree of λ is n matrix
- *Galois theory*: most polynomials of degree 5 or higher cannot be solved using *algebraic methods*.
- Oops - Approximate result
- No guarantees on computation time



Evariste Galois (1811-1832)

Matrix powers

The following works for both integer and fractional powers

- Let's compute

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$$\mathbf{A}^2 =$$

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- That includes the inverse as well:

$$\mathbf{A}^{-1} = \mathbf{V} \mathbf{\Lambda}^{-1} \mathbf{V}^{-1}$$

Observation when taking high powers of a matrix

(Building on the power identity from the previous slide)

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$$\mathbf{A}^k = \mathbf{V} \mathbf{\Lambda}^k \mathbf{V}^{-1}$$

$$\mathbf{\Lambda}^k = \begin{pmatrix} \lambda_1^k & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix}$$

As $k \rightarrow \infty$, first entry becomes **huge** compared to others.

Observation when taking high powers of a matrix

(Building on the power identity from the previous slide)

$$\mathbf{A}^k = \mathbf{V} \mathbf{\Lambda}^k \mathbf{V}^{-1}$$

$$\mathbf{\Lambda}^k \approx \begin{pmatrix} \lambda_1^k & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

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Power iteration

A simple method to compute the dominant eigenvector

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Power iteration:

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x = np.random.random(size)

for i in range(num_iterations):
    x = A @ x
    x /= np.linalg.norm(x)
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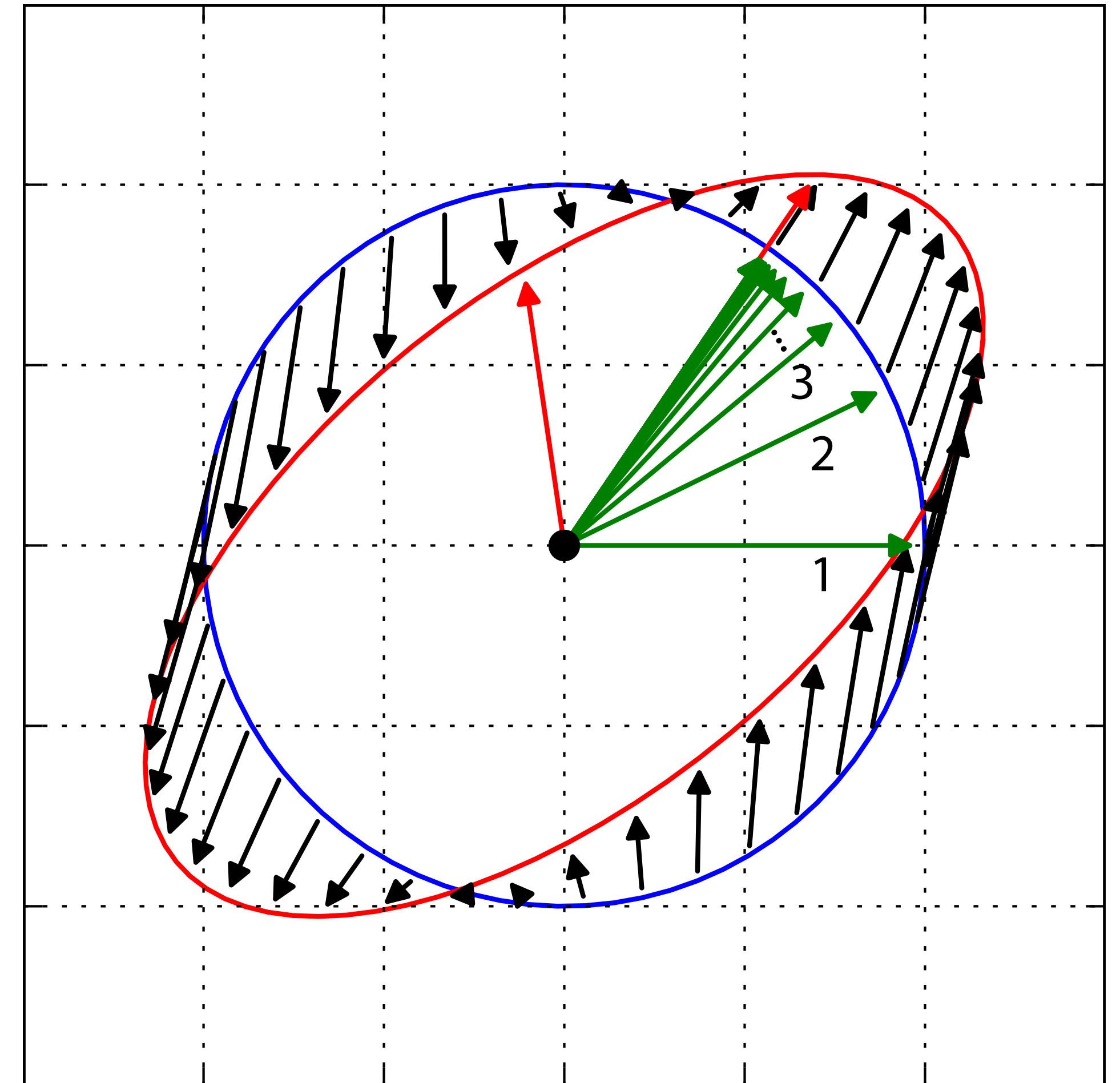
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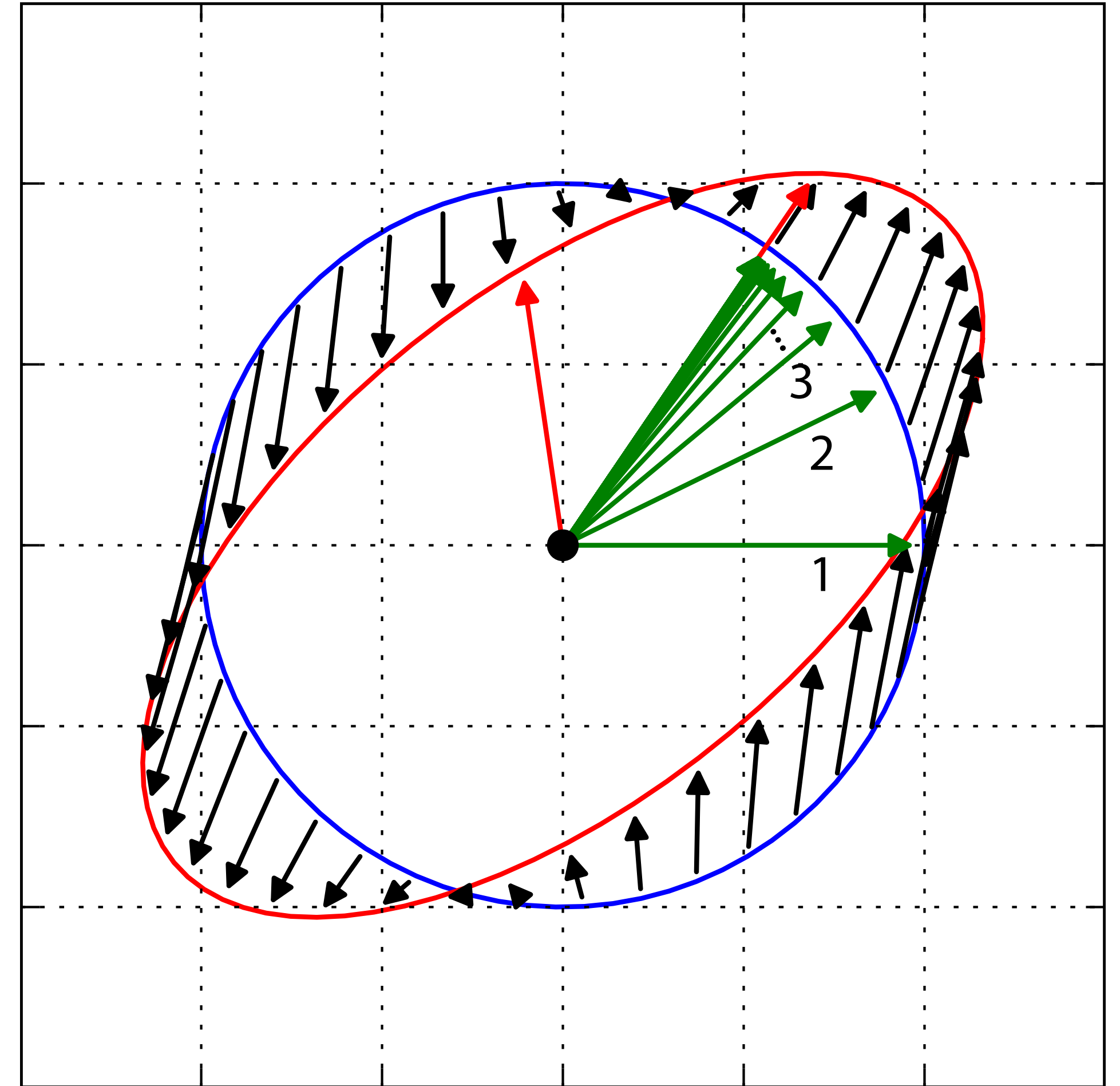
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Inverse iteration:

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A_lup = la.lu_factor(A)

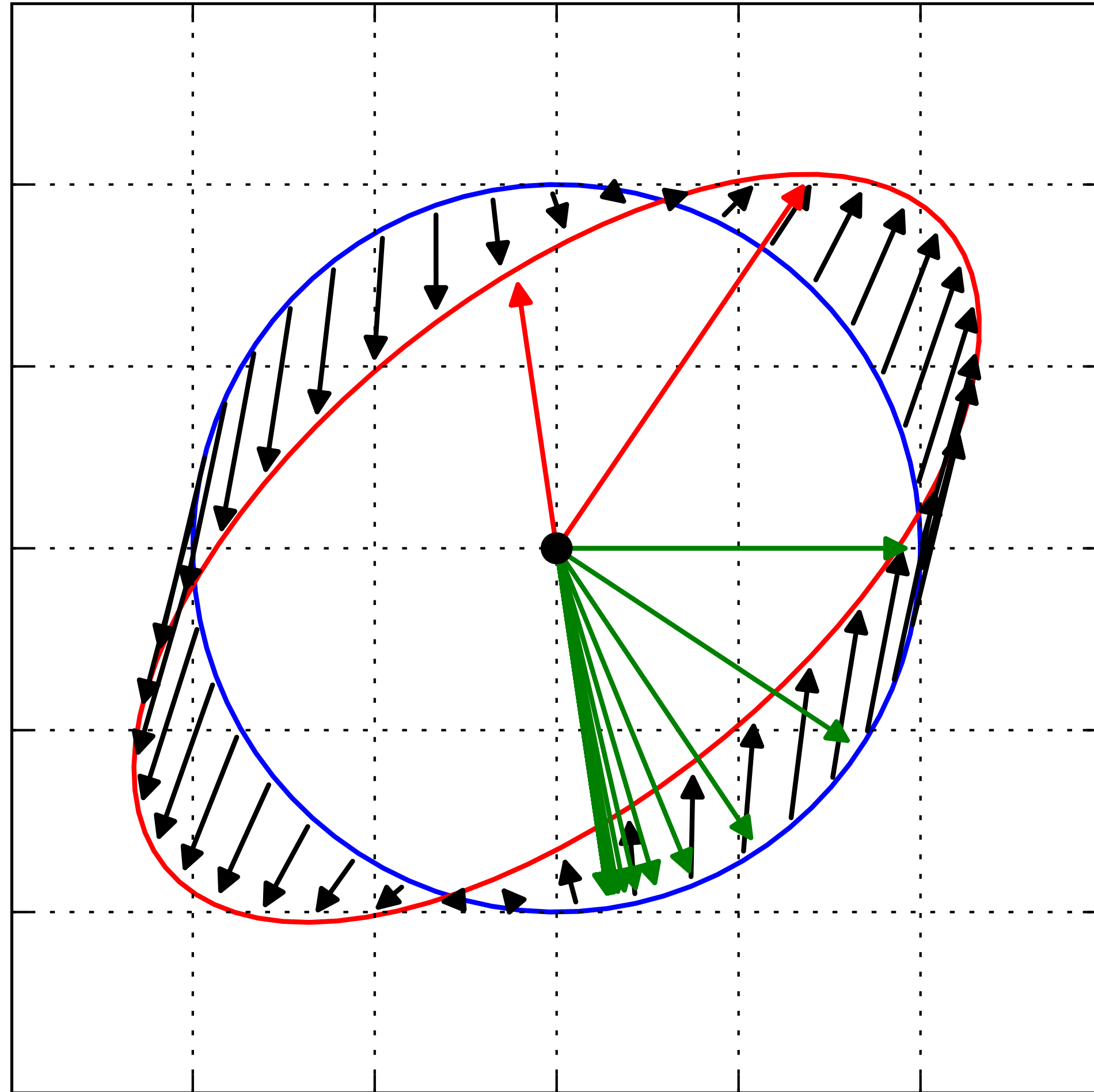
for i in range(num_iterations):
    x = la.lu_solve(A_lup, x)
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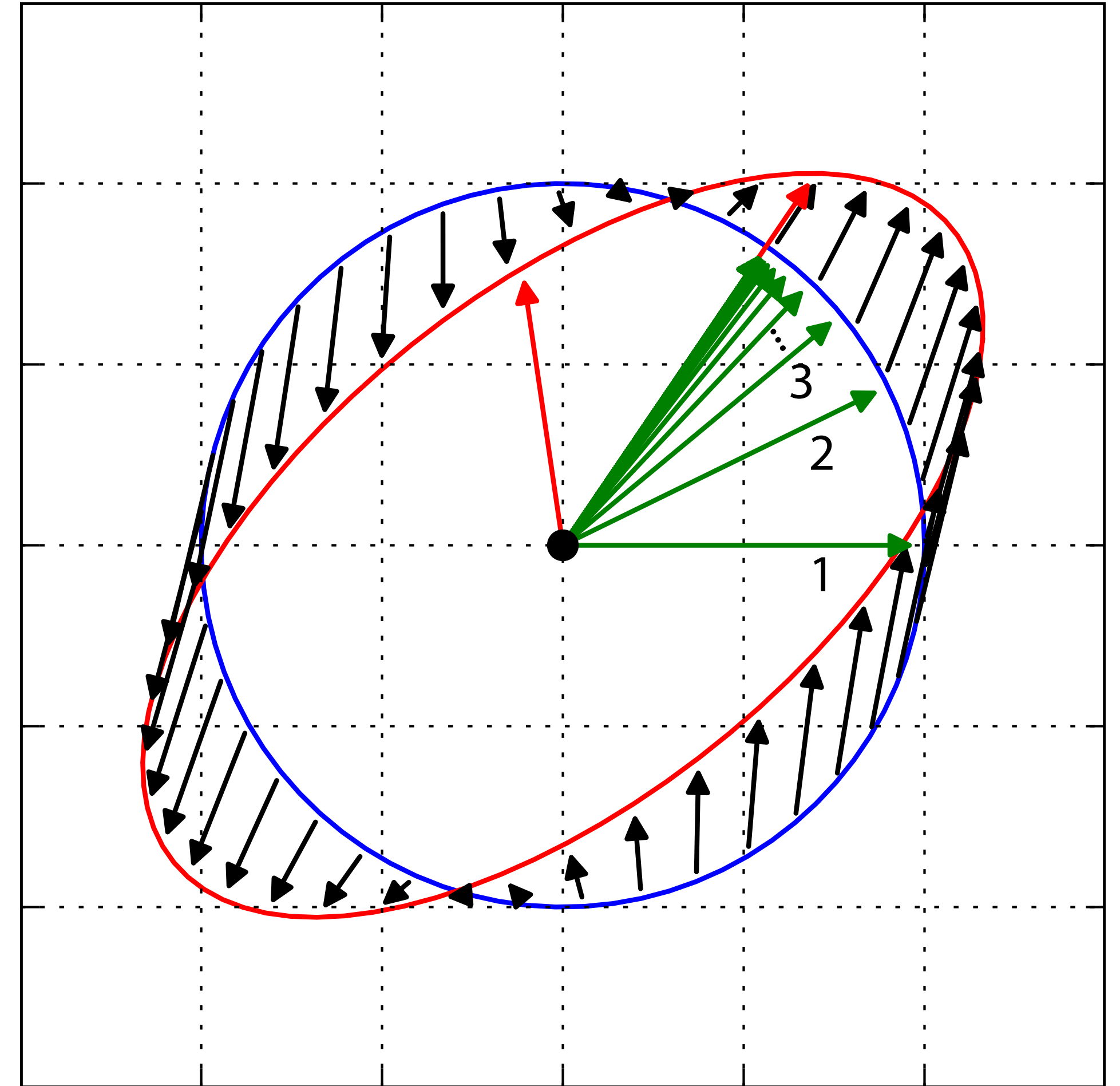
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QR algorithm

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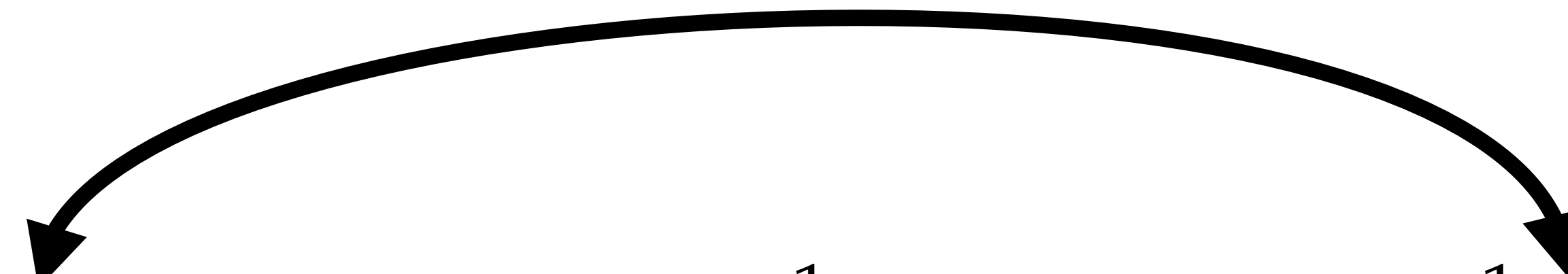
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QR algorithm

Similarity transformation: Same eigenvalues

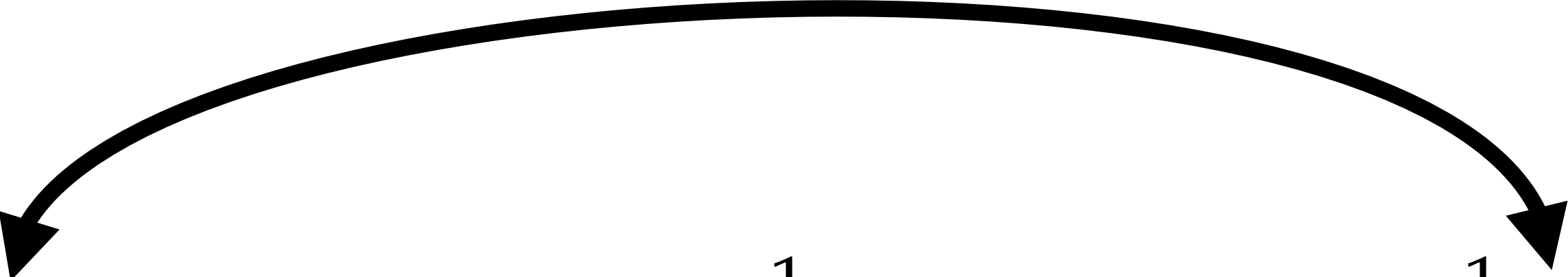
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$$\mathbf{A}_{k+1} = \mathbf{R}_k \mathbf{Q}_k = \mathbf{Q}_k^{-1} \mathbf{Q}_k \mathbf{R}_k \mathbf{Q}_k = \mathbf{Q}_k^{-1} \mathbf{A}_k \mathbf{Q}_k$$

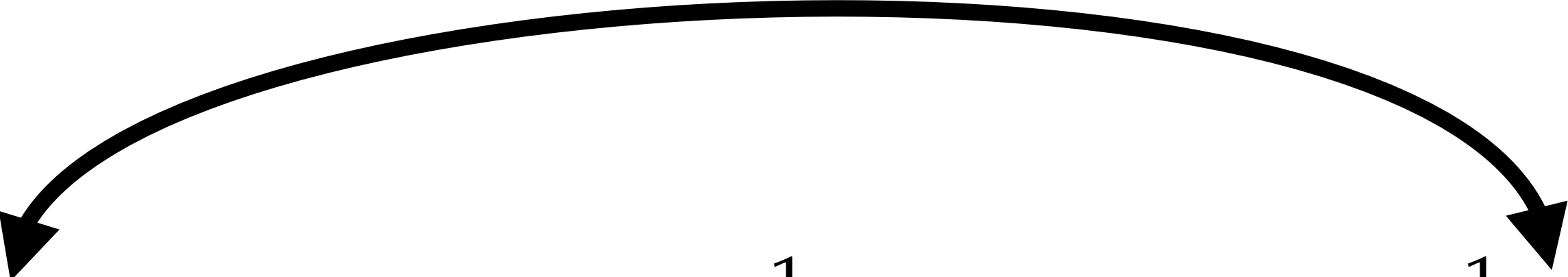
- **Intuition:** Run n instances of the power iteration at the same time.
 - QR factorization renormalizes vectors and keeps the eigenvector estimates from all collapsing onto the same solution.

Demo time

QR algorithm

Similarity transformation: Same eigenvalues

```
def eig(A, n_it=10):  
    for i in range(n_it):  
        Q, R = la.qr(A)  
        A = R @ Q  
    return A
```

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 - QR factorization renormalizes vectors and keeps the eigenvector estimates from all collapsing onto the same solution.
- In practice: many subtleties, use libraries and don't build your own version.

Application: Solving linear ODEs

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A way to evaluate $\exp(A)$ when A is a matrix.

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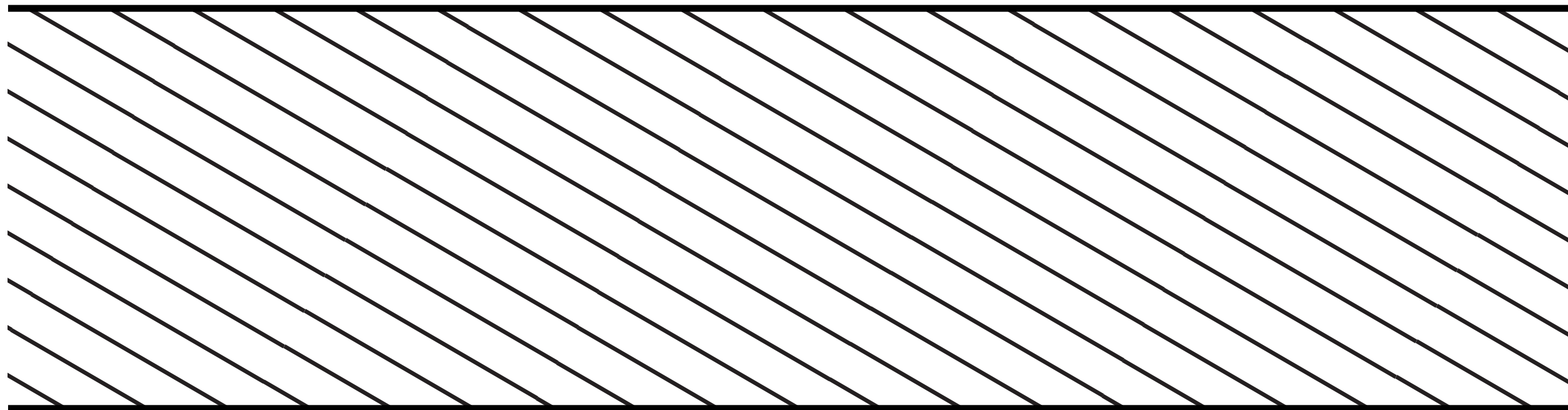
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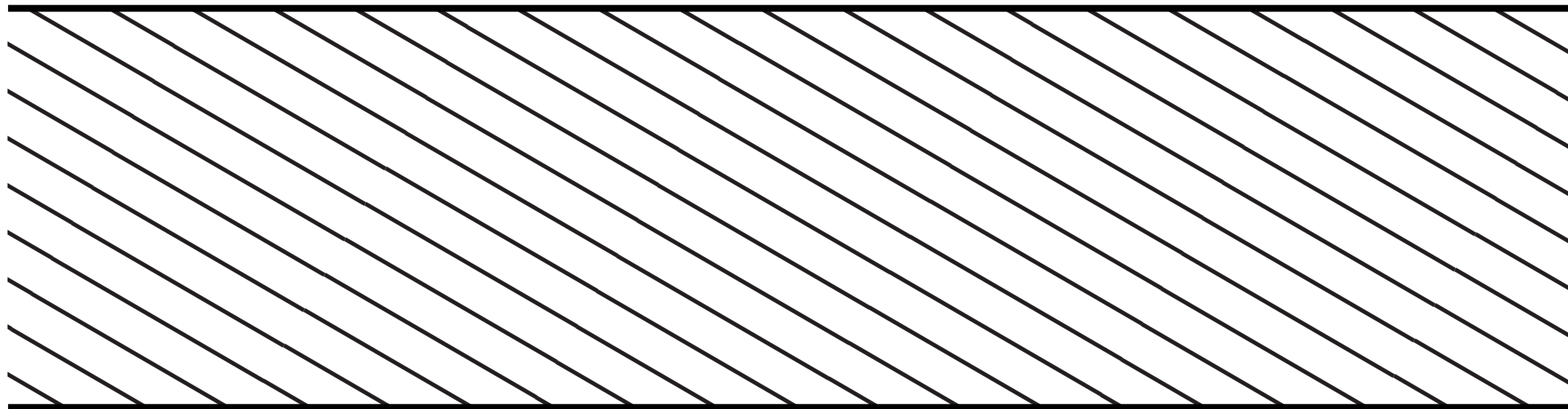
A random example from my own research

Scattering in dusty layers



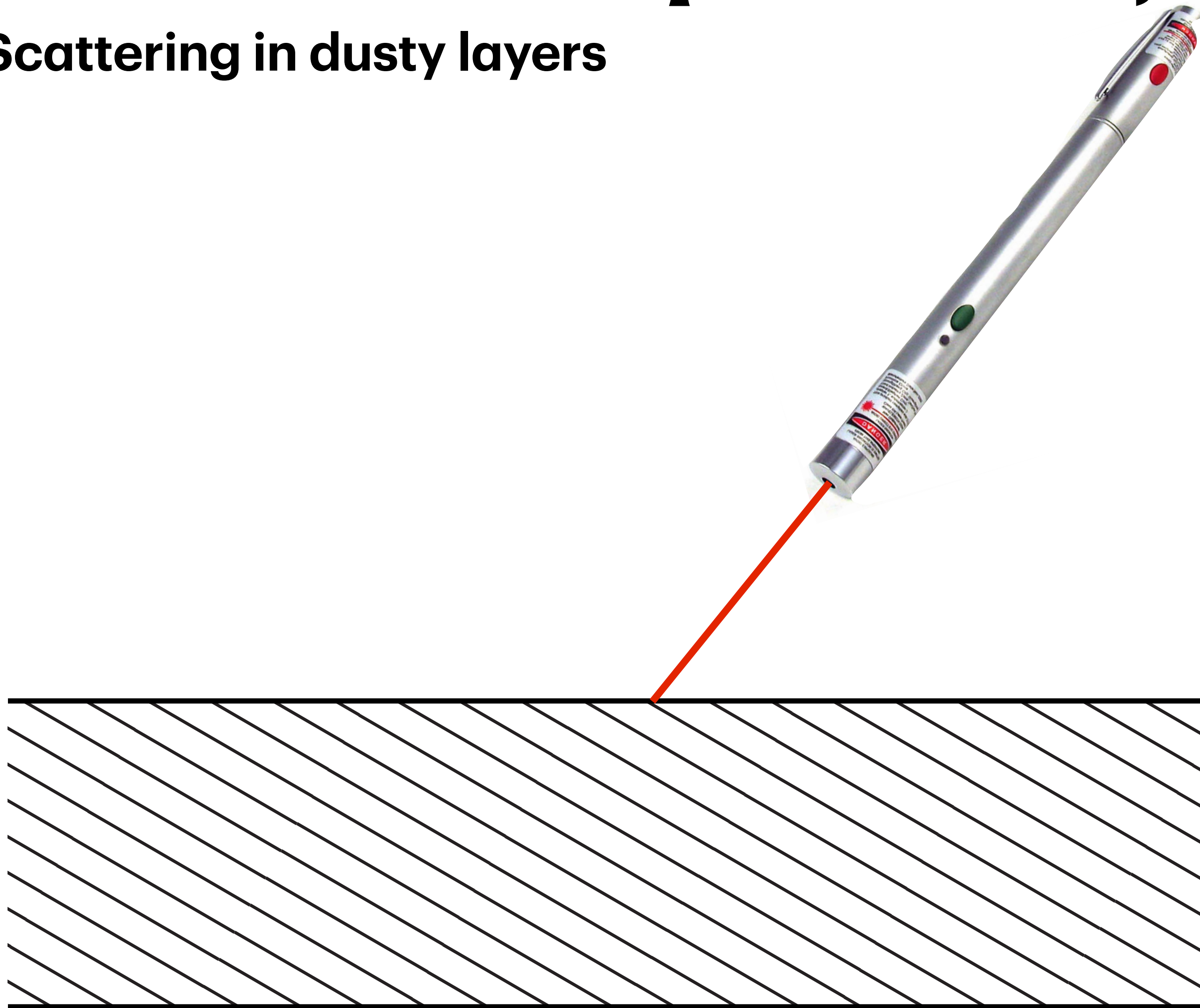
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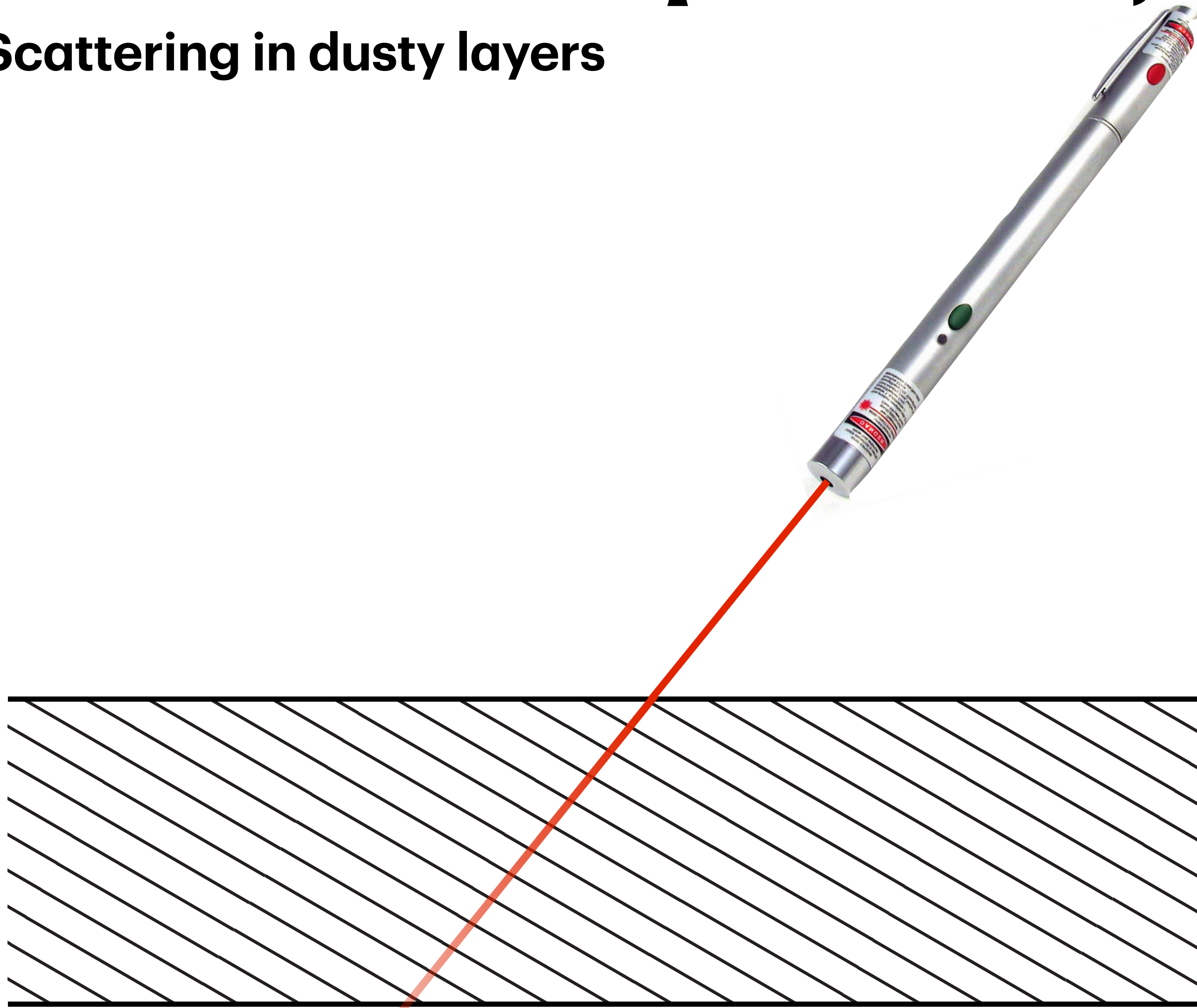
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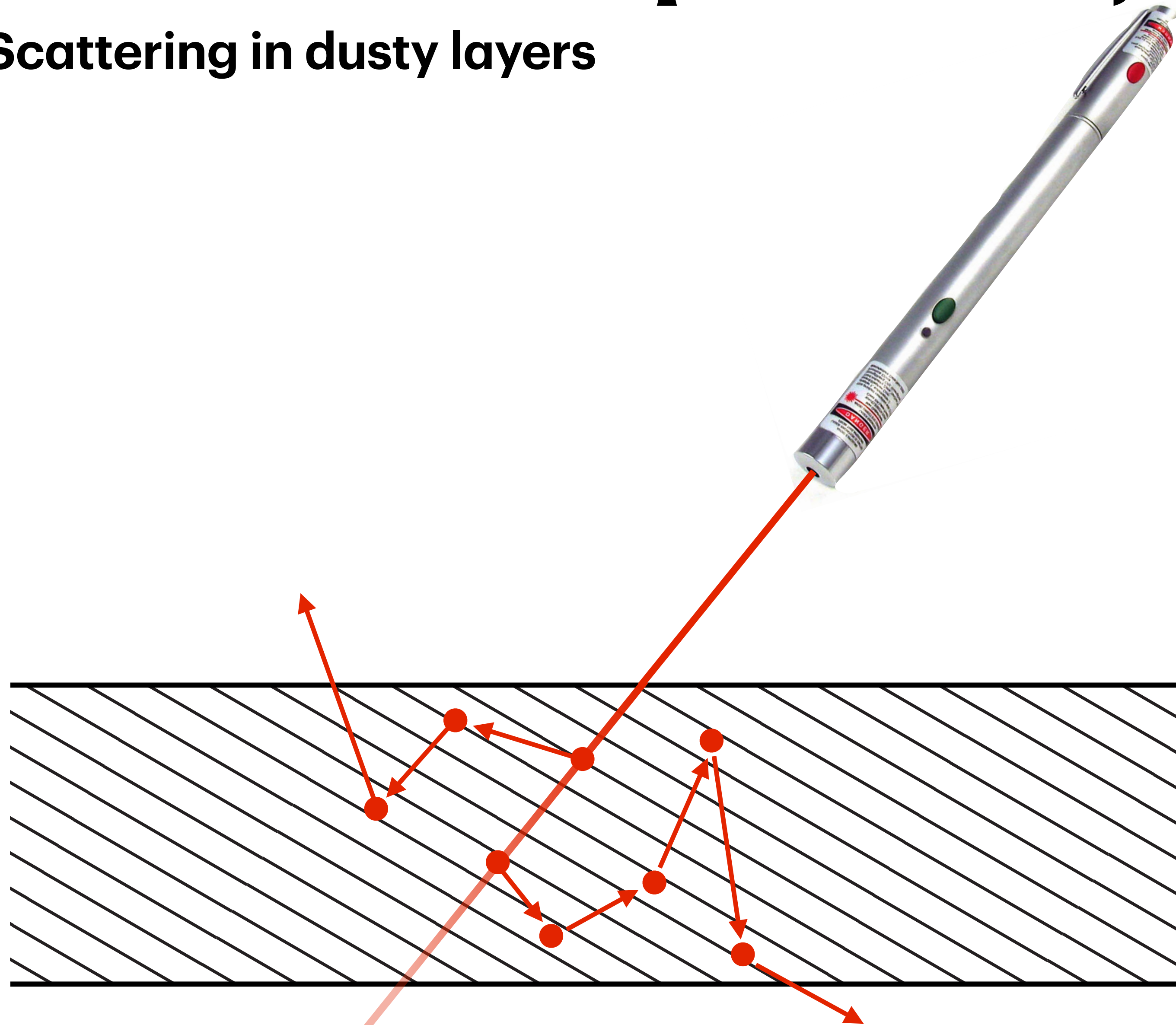
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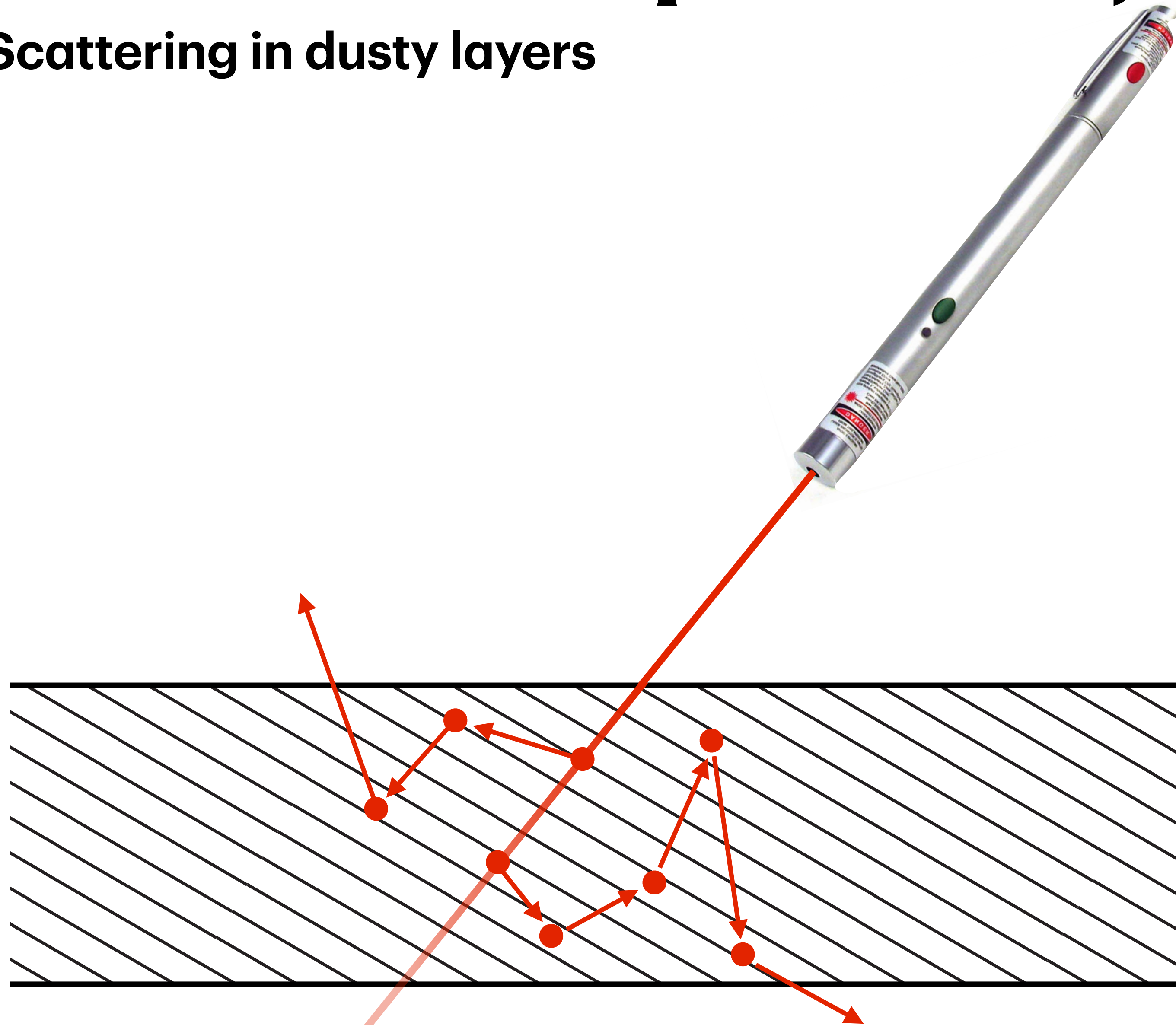
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Problem has the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

$\mathbf{x}(t)$ is the light traveling in different directions.

t is the depth within the layer.



$\alpha=0.02$

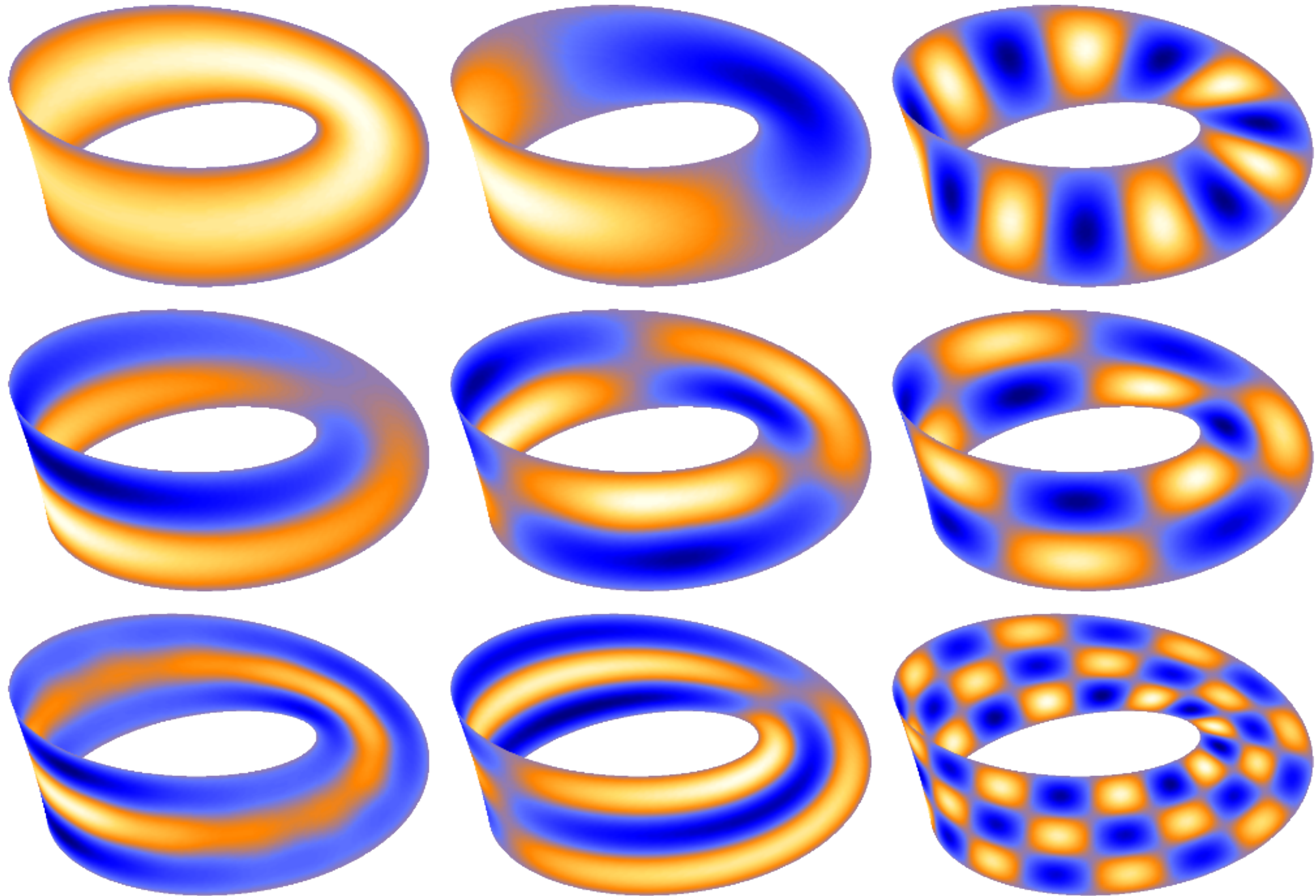
$\tau=0.000, g=0.900$



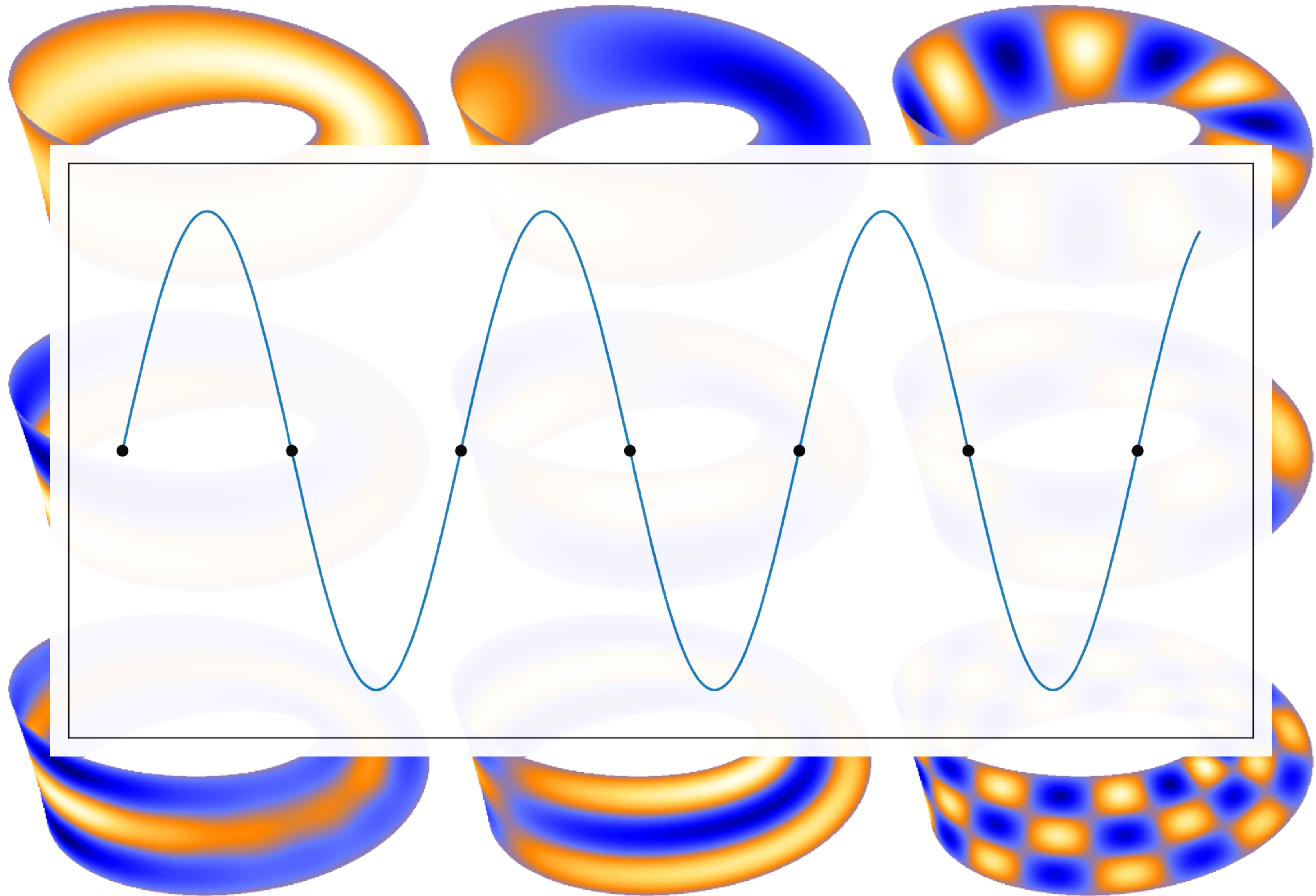
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Oscillations on surfaces



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[Langlois, An, Jin, and James, SIGGRAPH 2014]



Diameter: 19 cm

Vertices: 51434

Compressed: 26 kB

Uncompressed: 28 MB

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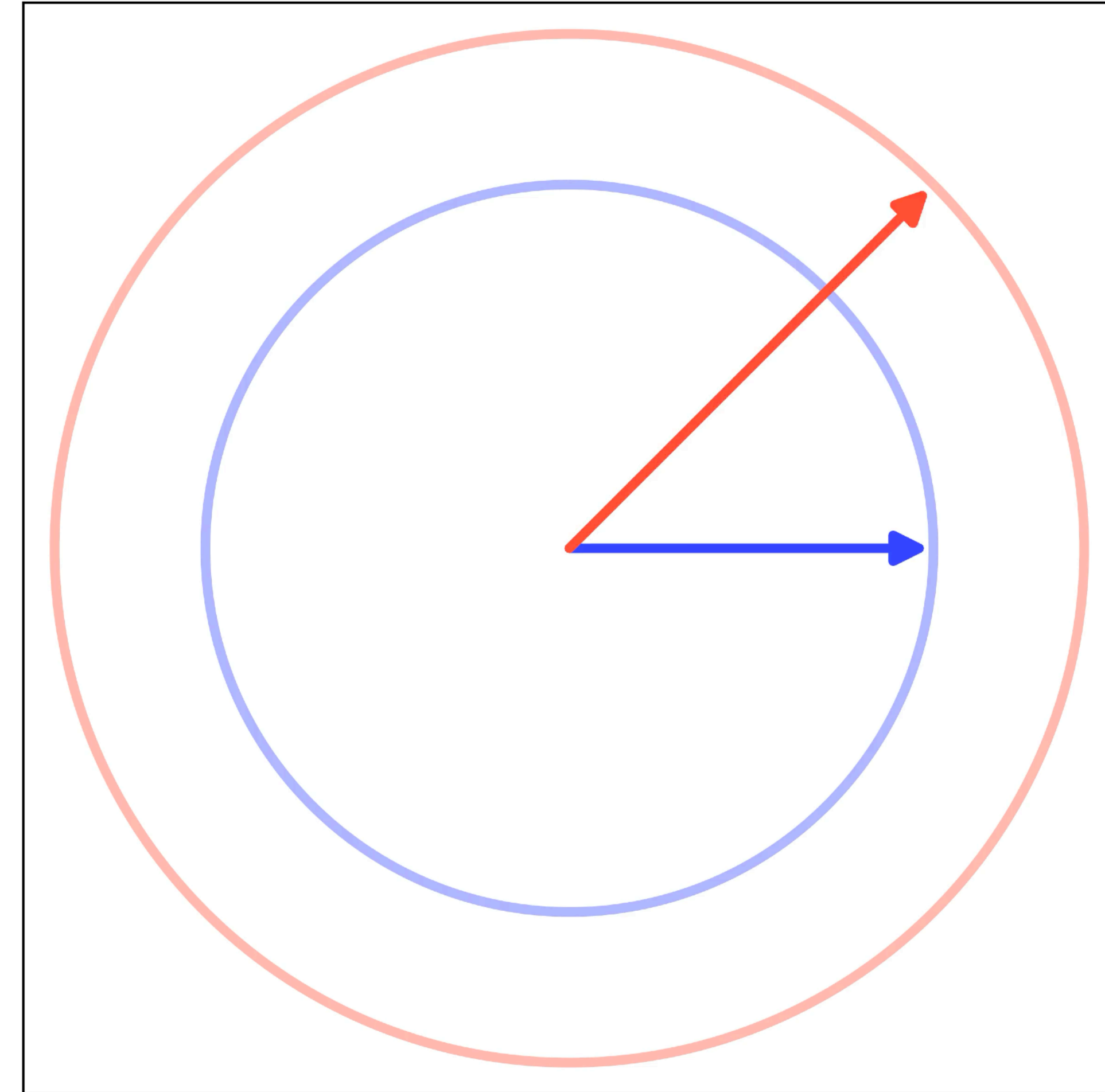
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Eigenanalysis in practice: it's complicated..

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where

$$\mathbf{y}^T \mathbf{A} = \lambda \mathbf{y}^T \qquad \mathbf{A} \mathbf{x} = \lambda \mathbf{x}$$

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4. Doesn't make sense for non-square \mathbf{A} !

The Singular Value Decomposition

LU

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

QR

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} R \end{bmatrix}$$

SVD

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \Sigma \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

The Singular Value Decomposition



The Singular Value Decomposition

- Can be computed for **any** matrix (non-square, non-symmetric, singular, ..)
- Involves only orthogonal and diagonal matrices.
- **Impeccable** numerical properties.
- No complex arithmetic necessary.
- Expensive to compute (~5-10 times the cost of LU)
- SVD and Eigendecomposition are *identical** when **A** is symmetric.

* Except for tiny differences: ordering of entries, etc.

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Terminology

Let's look at the $m=n$ case first.

$$\mathbf{V} = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix}$$

"Right singular vectors"
Orthogonal.

$$\mathbf{U} = \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ | & & | \end{pmatrix}$$

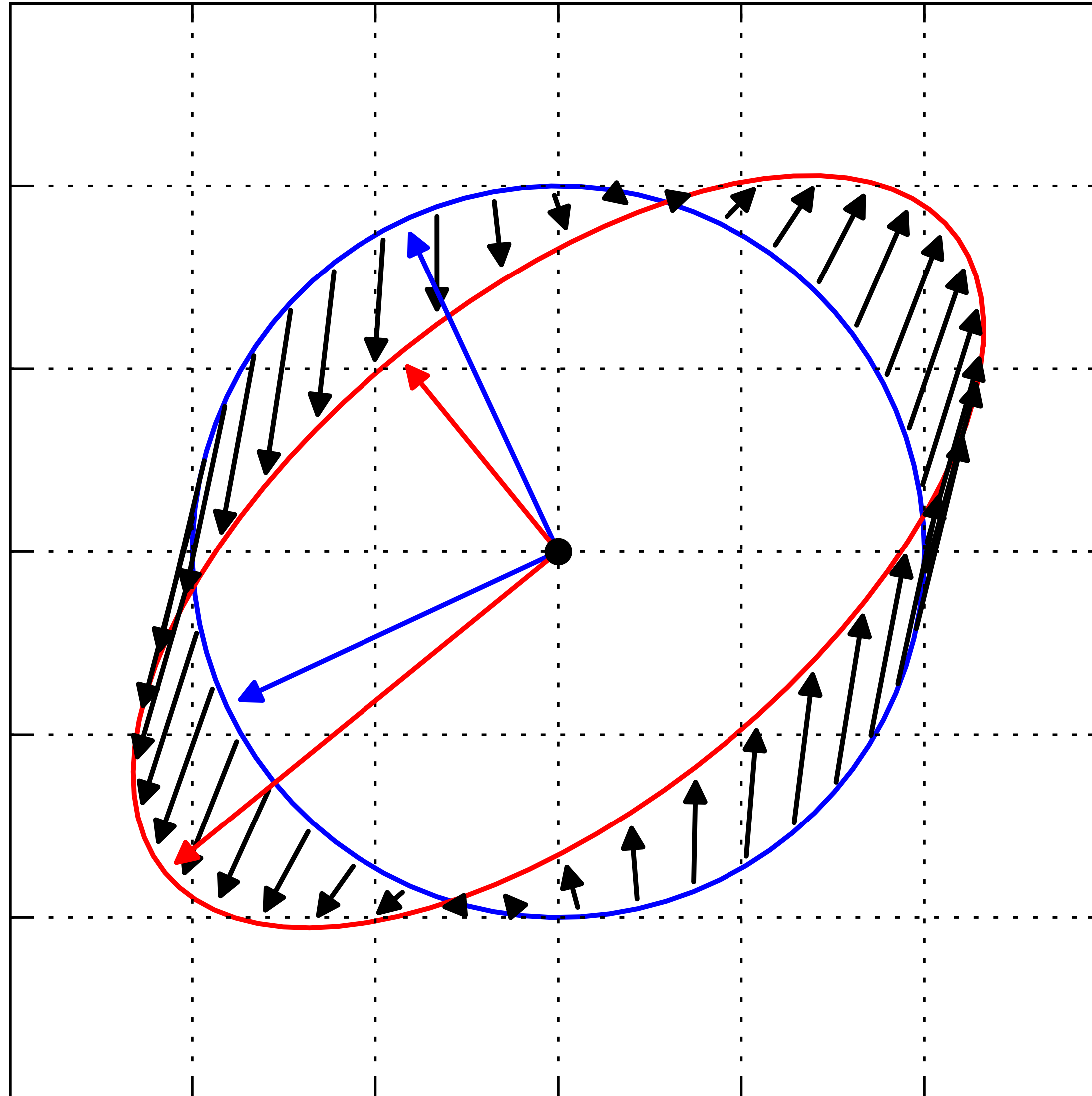
"Left singular vectors"
Orthogonal.

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

"Singular values"
Positive, sorted in
decreasing order

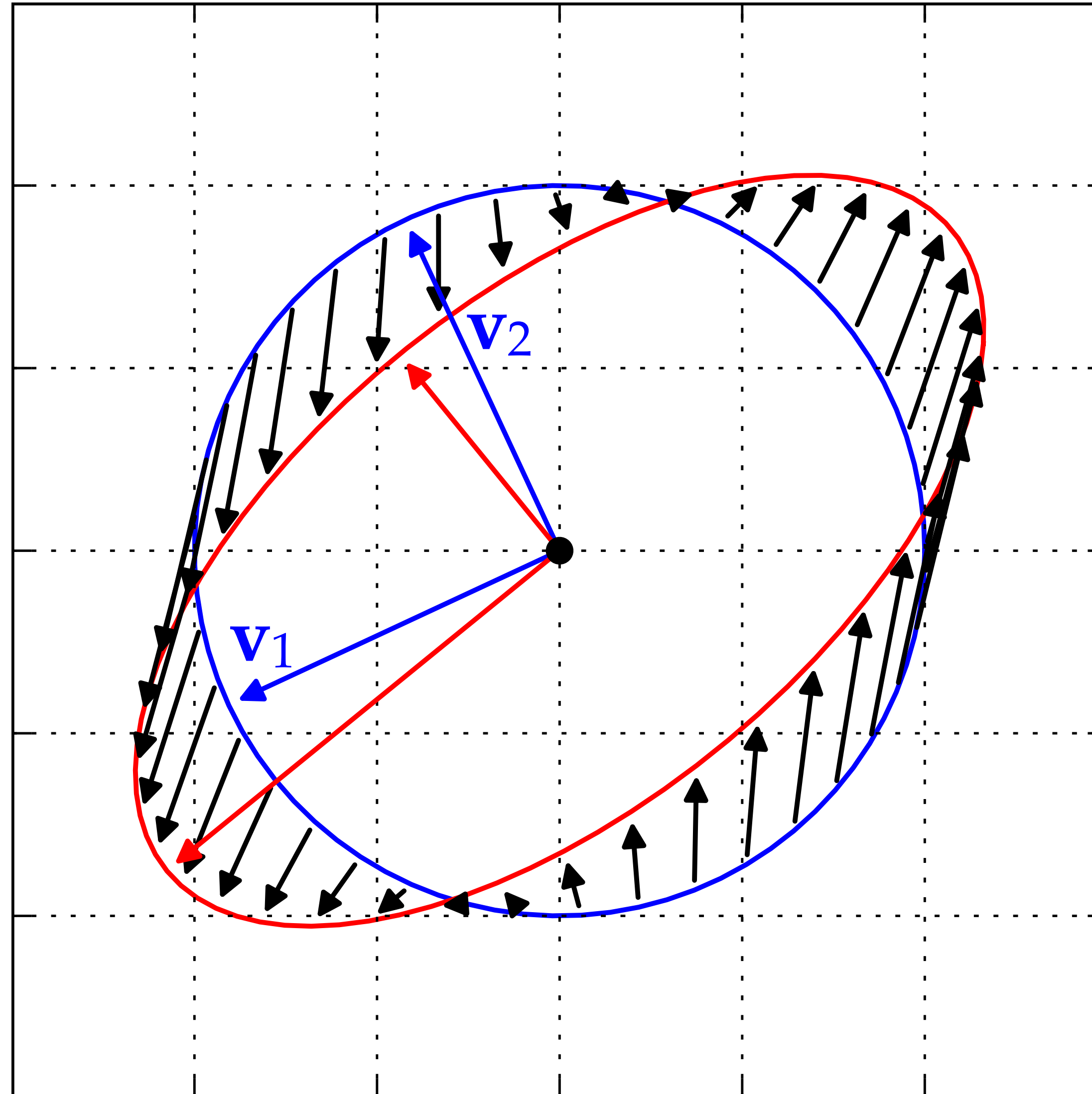
SVD

```
U,  $\Sigma$ , V = scipy.linalg.svd(A)
```



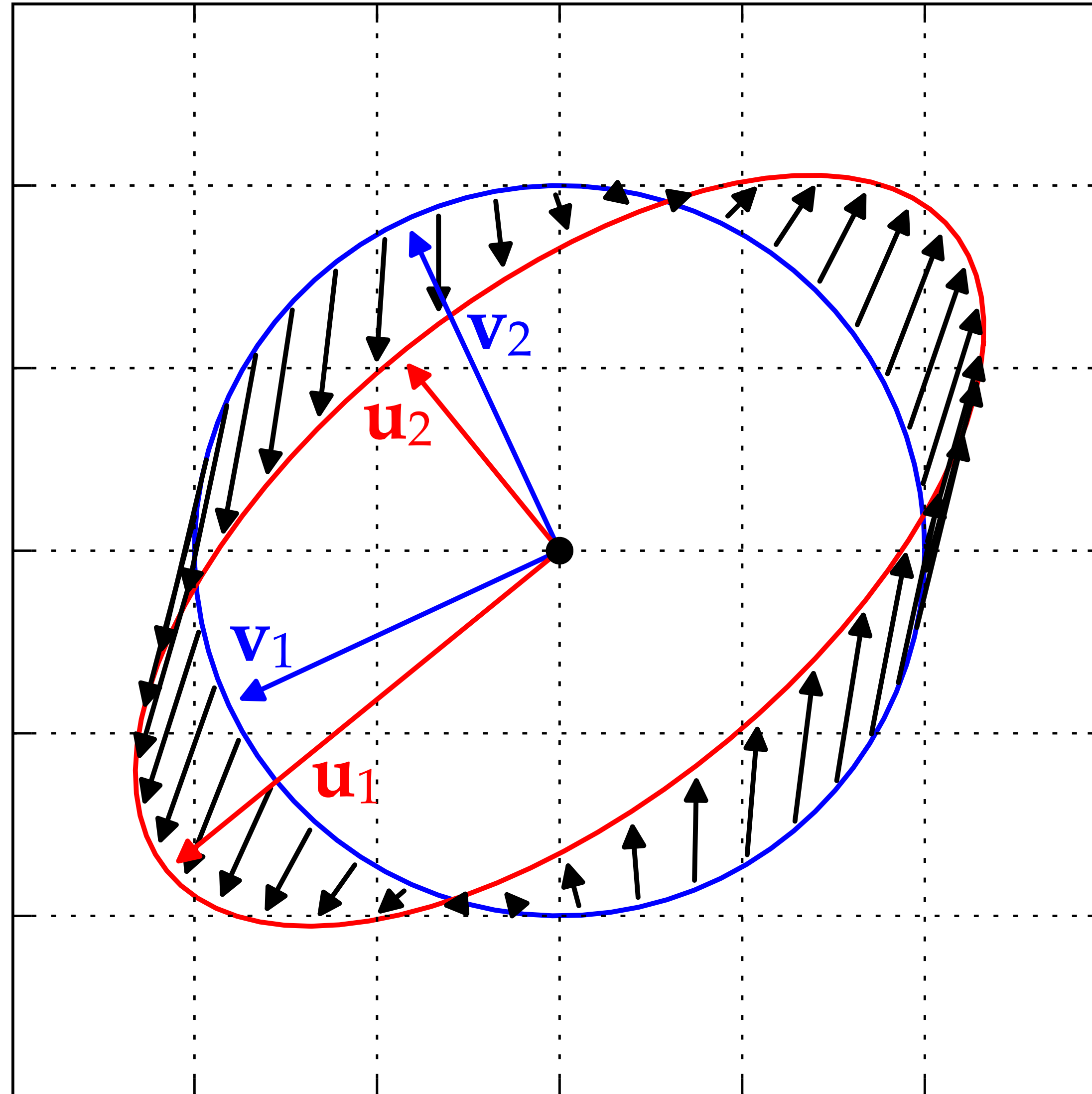
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Another view of the SVD

$$\begin{bmatrix} \mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{U} \end{bmatrix} \begin{bmatrix} \Sigma \end{bmatrix} \begin{bmatrix} \mathbf{V}^T \end{bmatrix}$$

Another view of the SVD

$$\boxed{\mathbf{A}} = \boxed{\mathbf{U}} \boxed{\Sigma} \boxed{\mathbf{V}^T}$$

where

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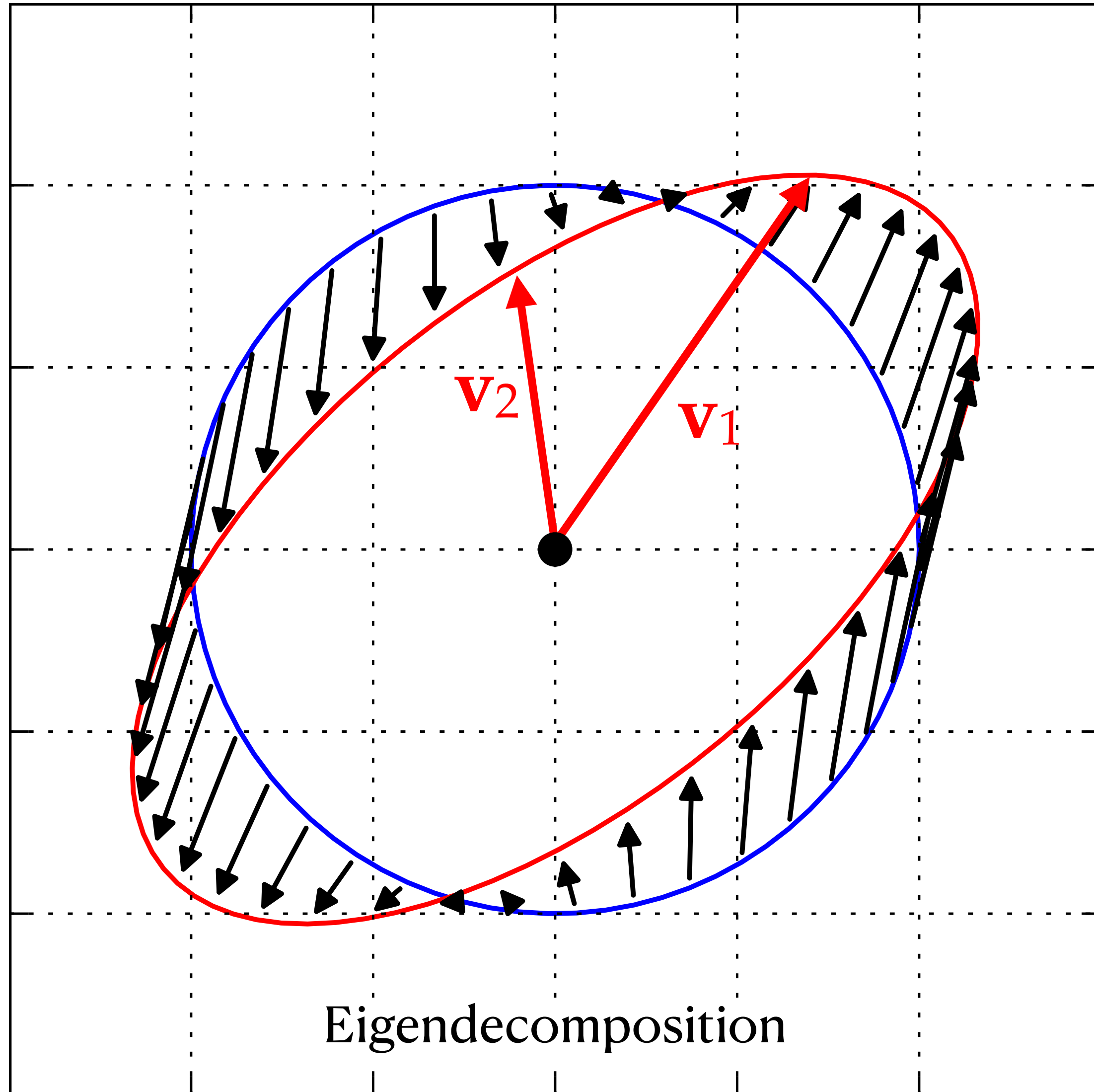
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$$\text{Then } \mathbf{A} = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

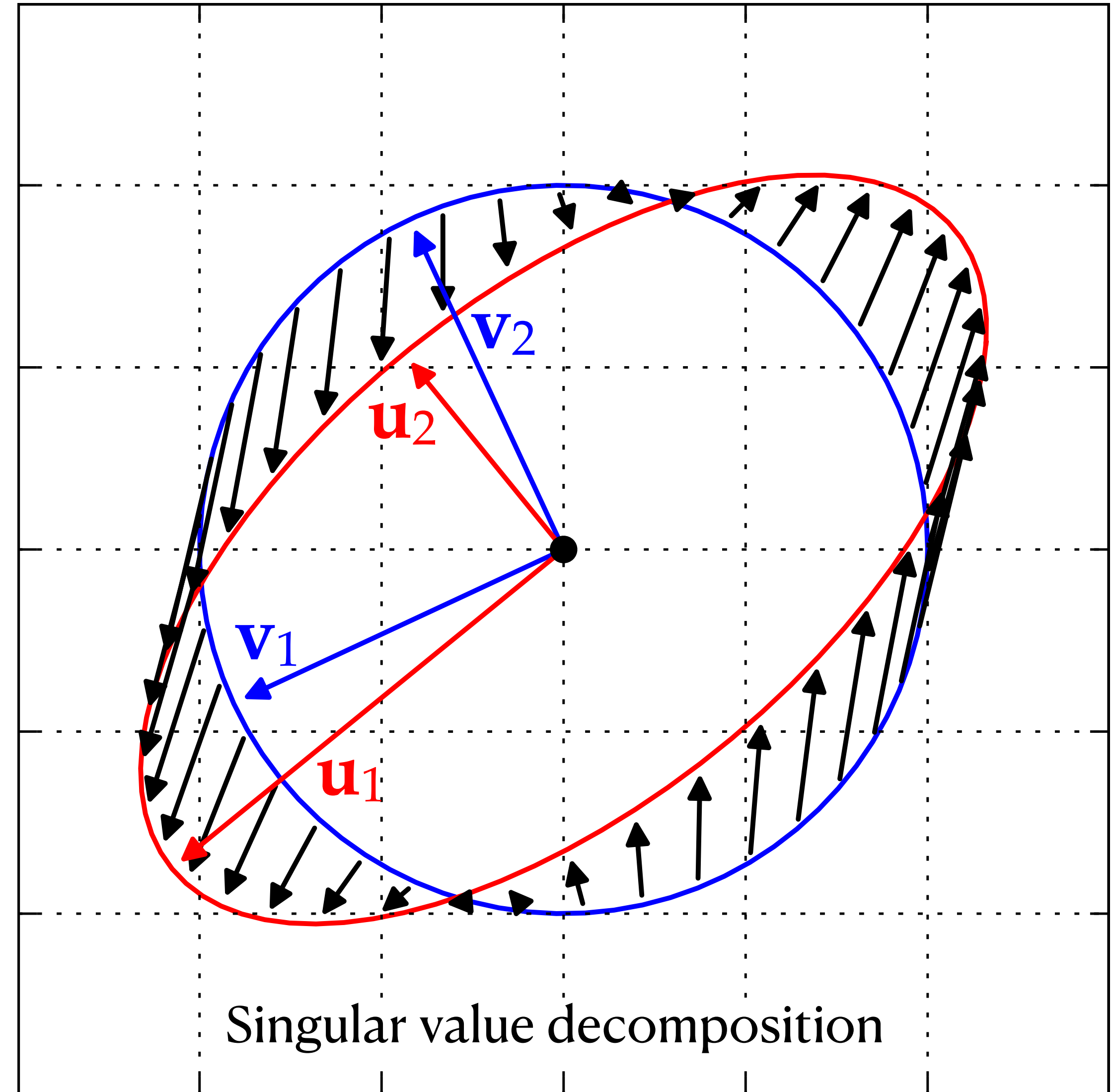
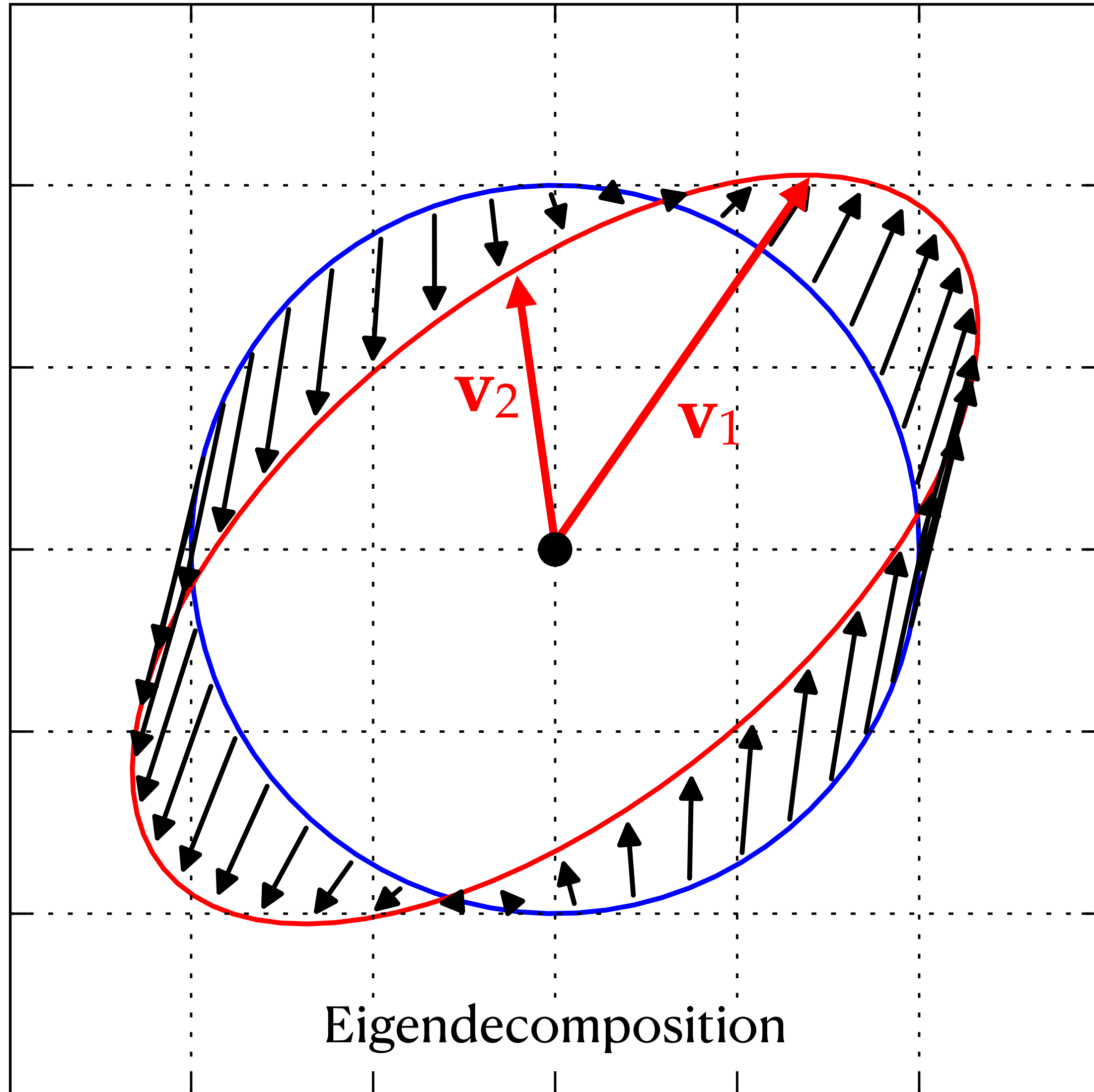
SVD vs Eigendecompositions

- A linear transformation maps a circle/sphere onto an ellipse/ellipsoid
- **Eigendecomposition**
 - Question: *which directions stay aligned when transforming by A ?*
- **SVD**
 - Question: *what are the principal axes of the ellipse/ellipsoid, and which directions on the circle map to them?*
- For symmetric matrices, this is the same question.

Eigendecomposition vs. SVD



Eigendecomposition vs. SVD



MATH-232 Review: The Normal Distribution

Univariate (1D) case. Also known as "*Gaussian distribution*"

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

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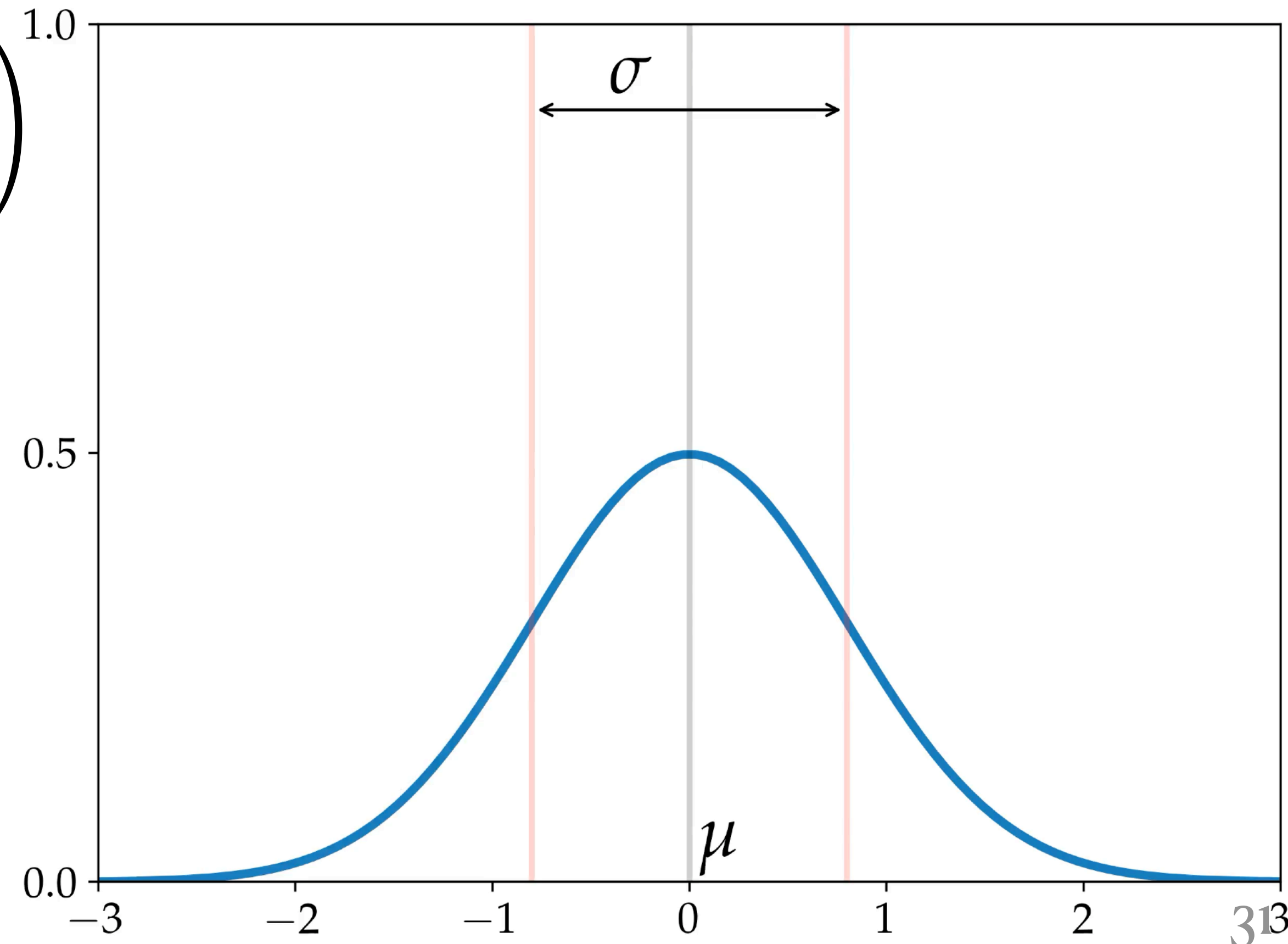
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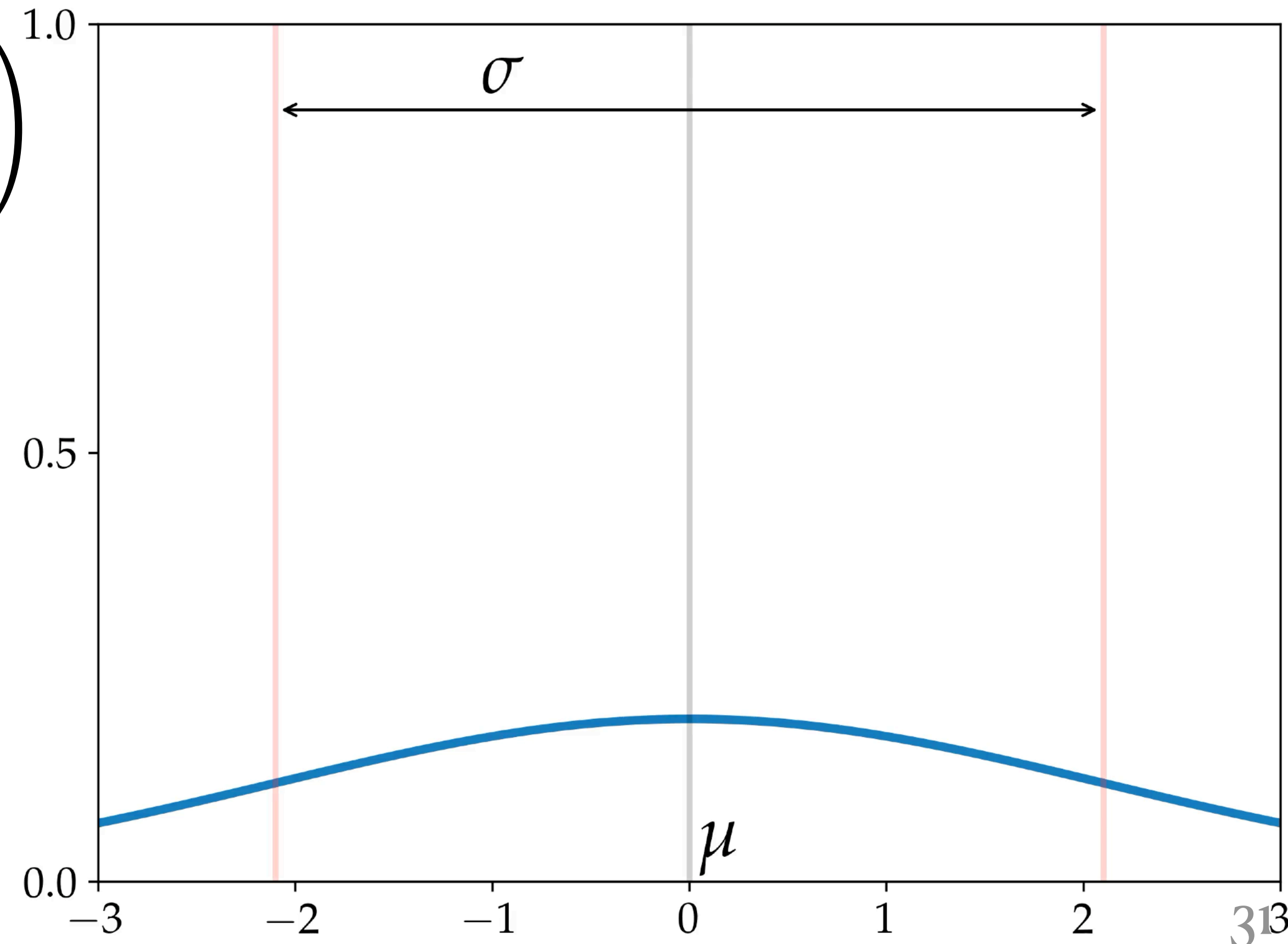
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Variance (square of std. deviation)

Standard deviation

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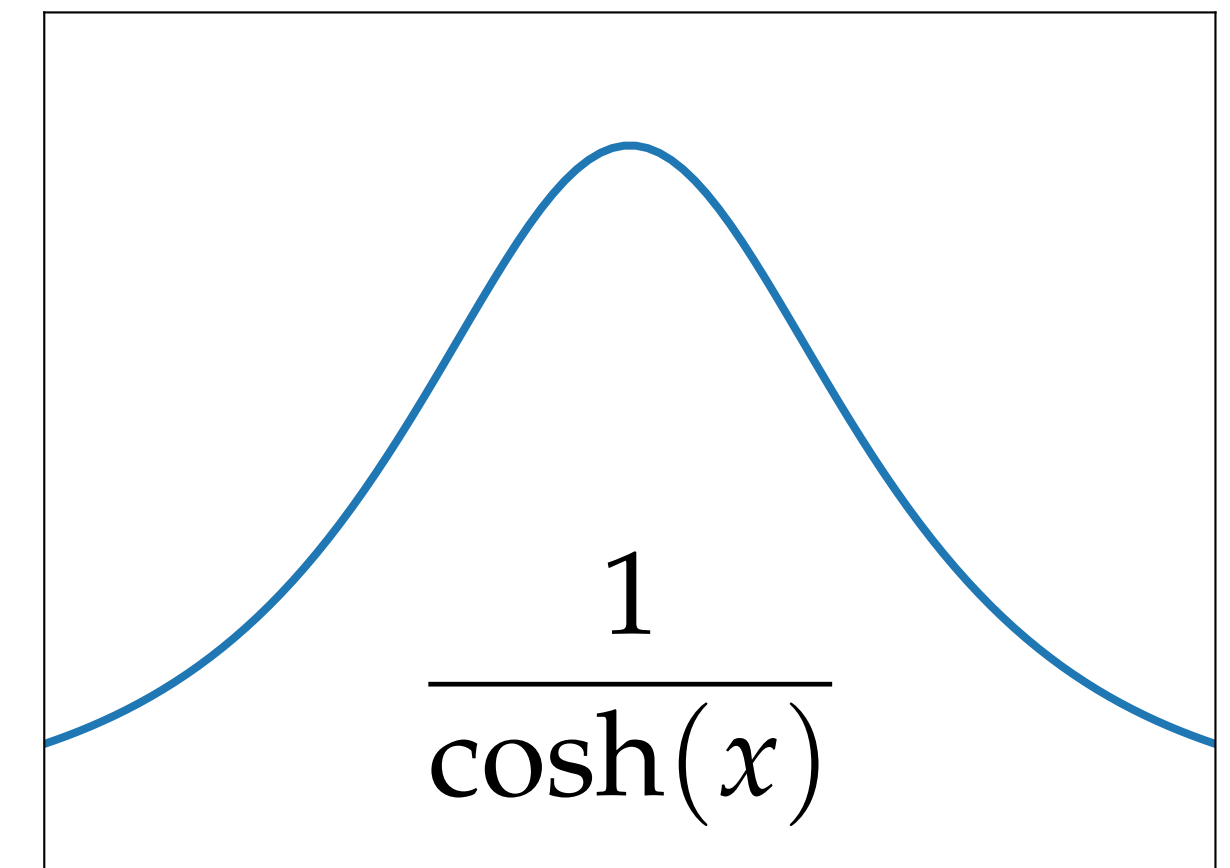
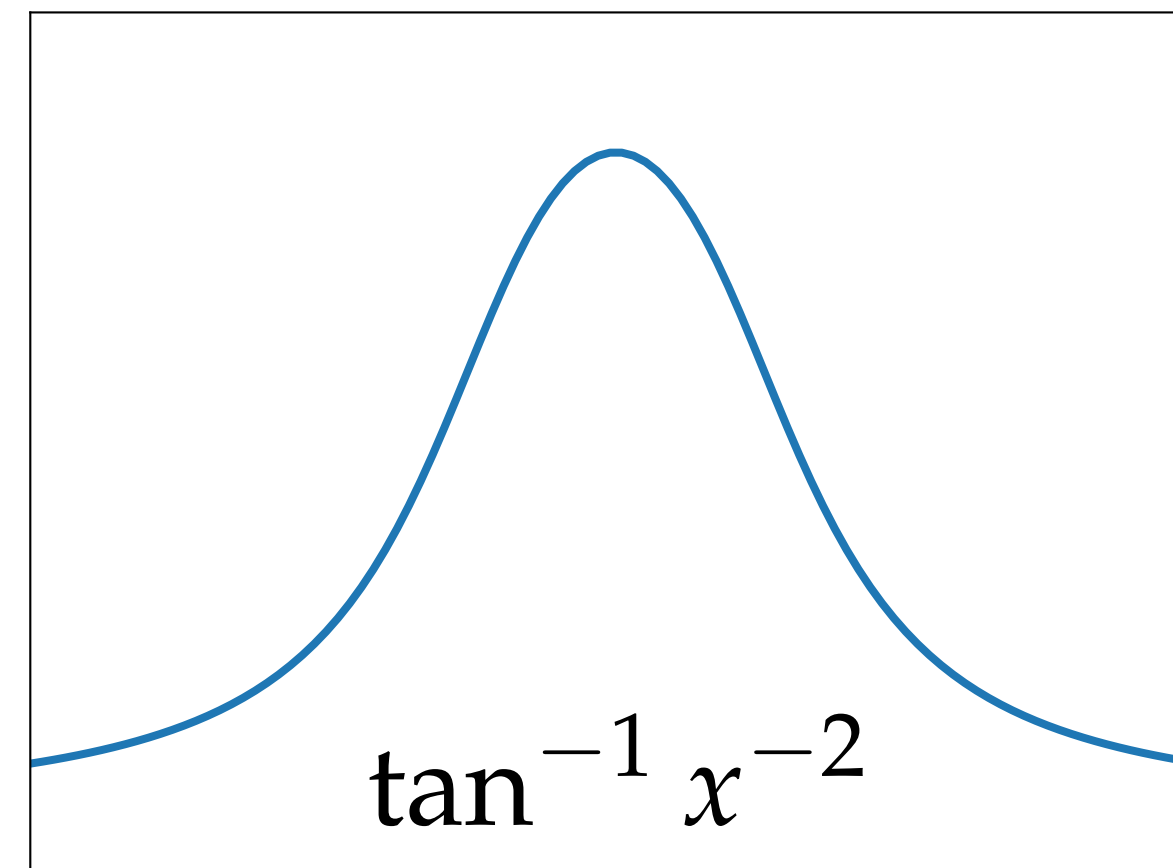
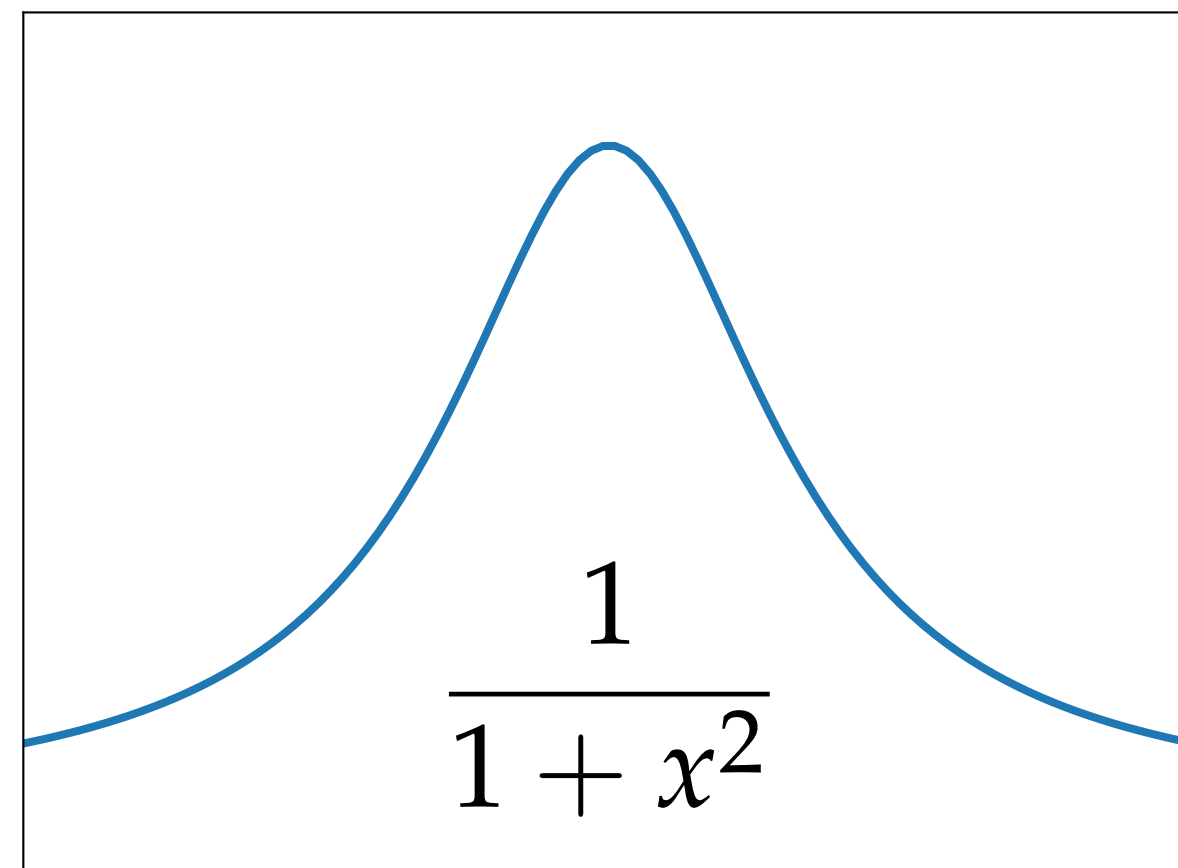
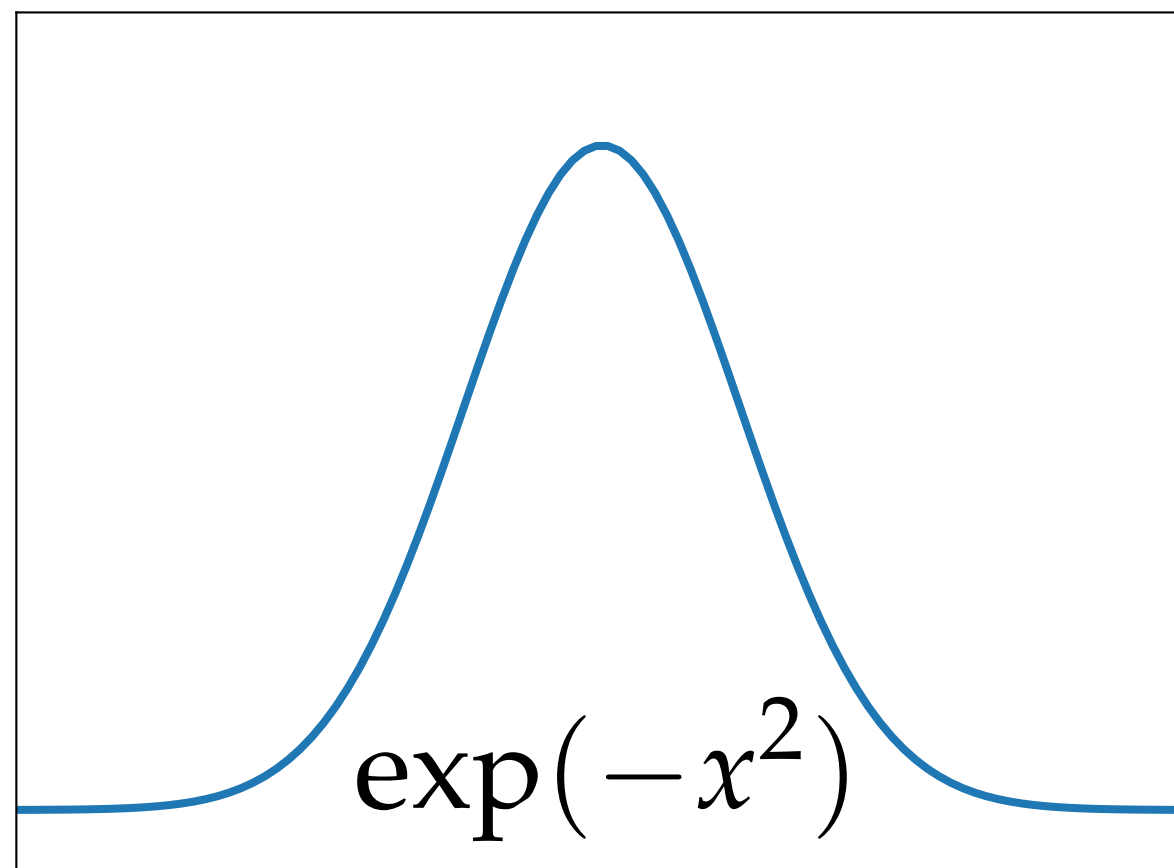
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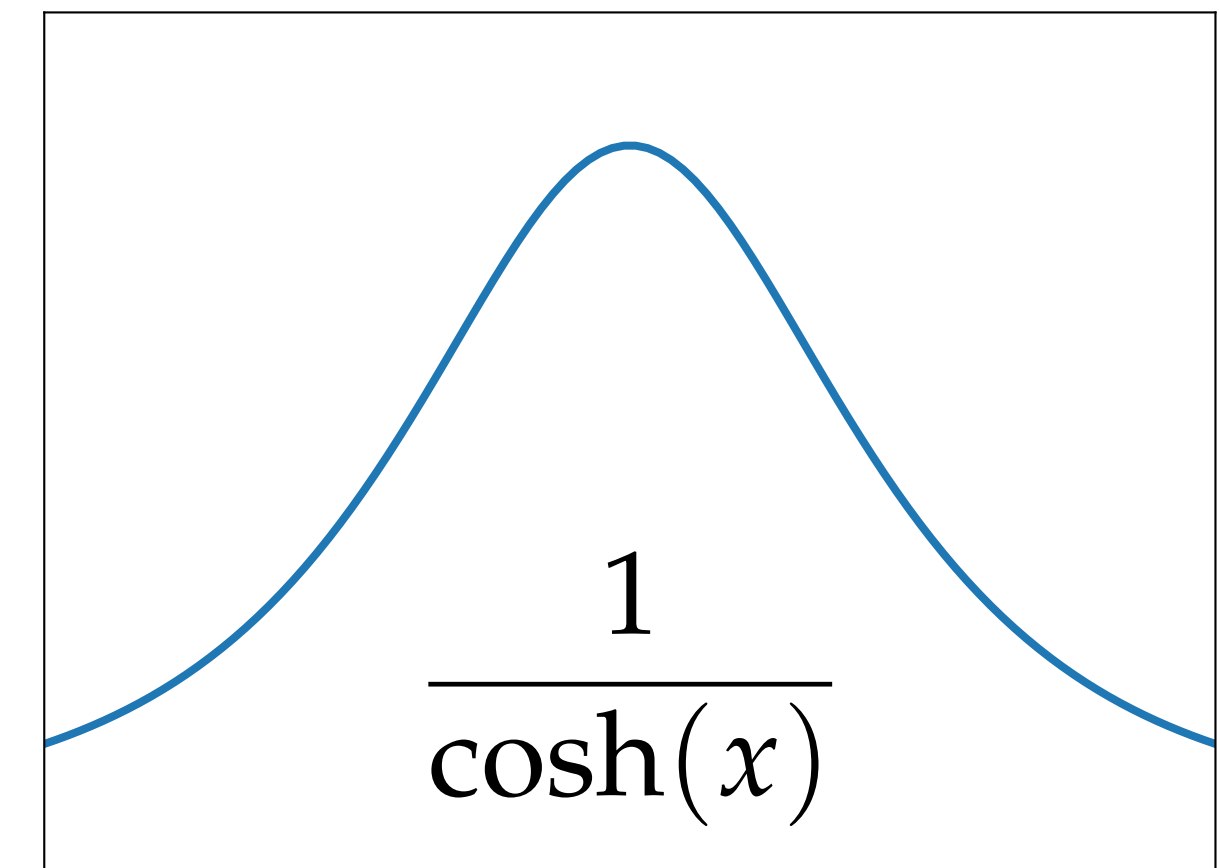
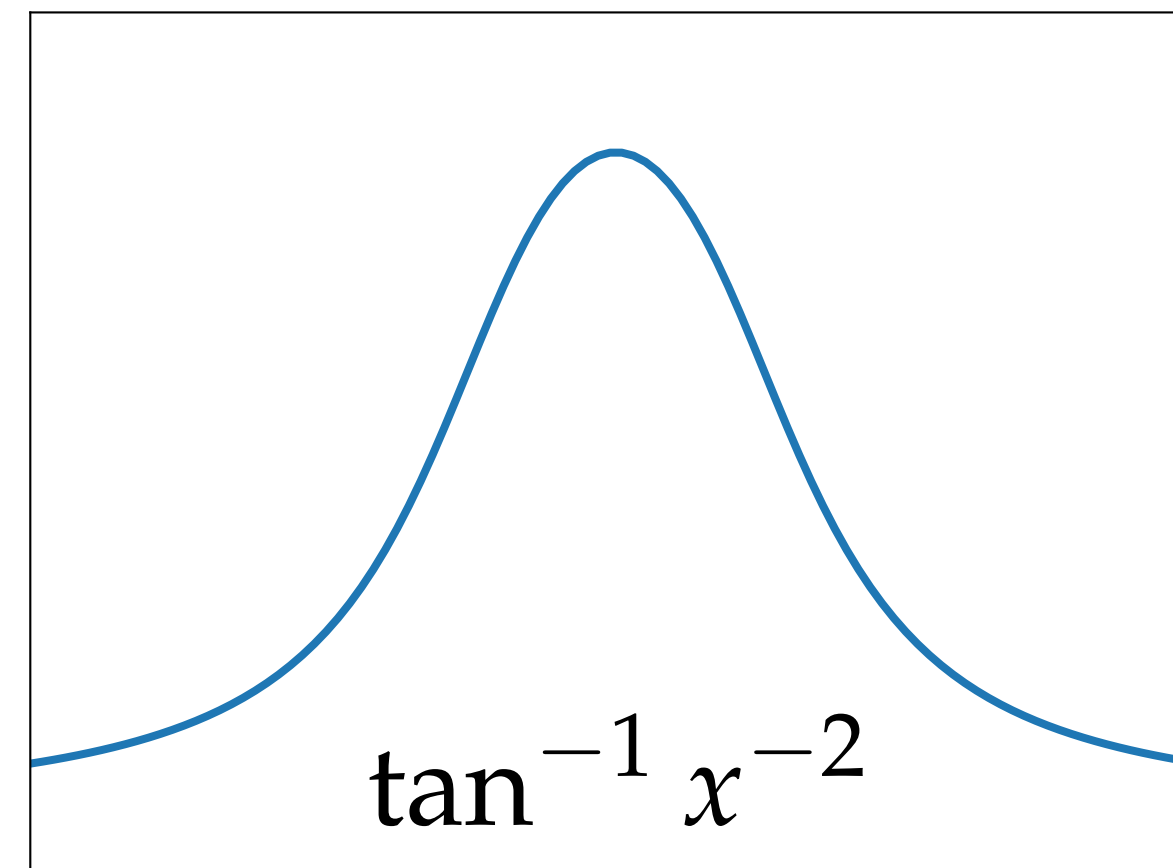
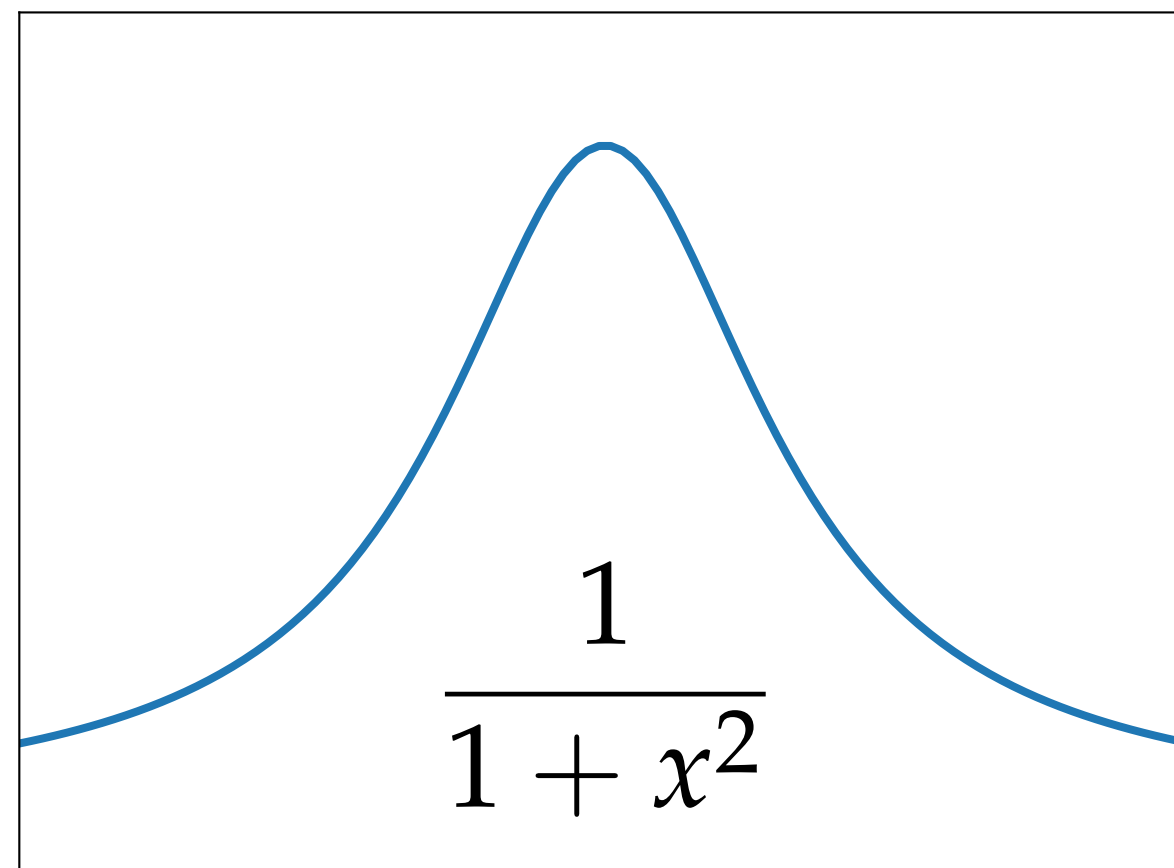
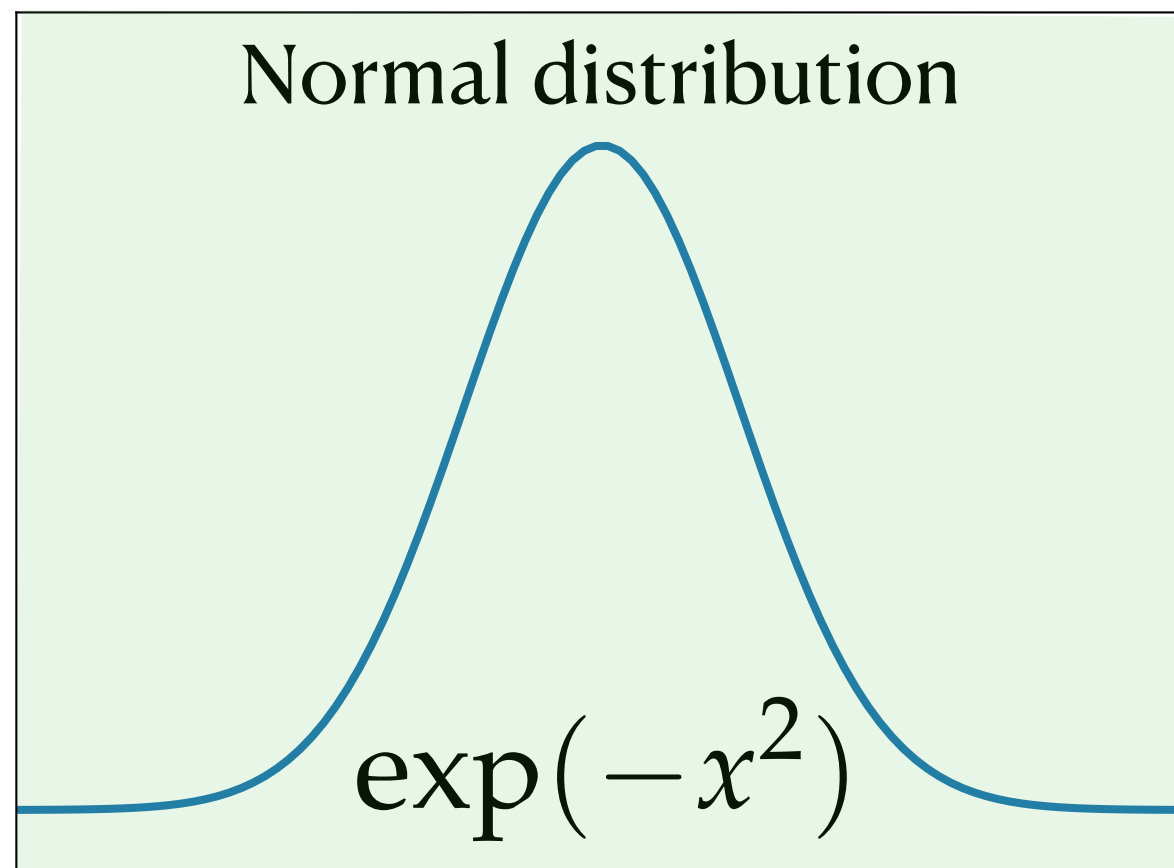
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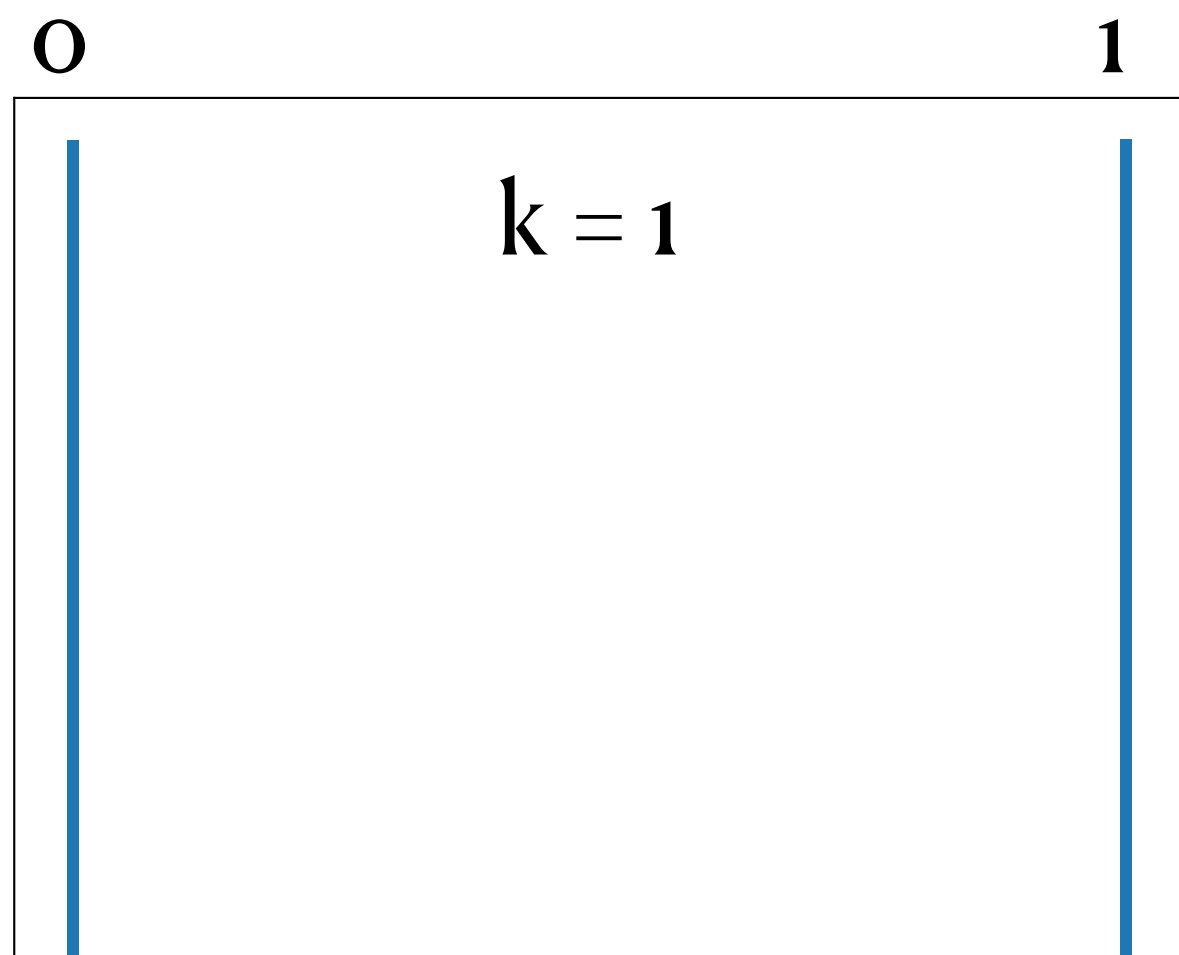
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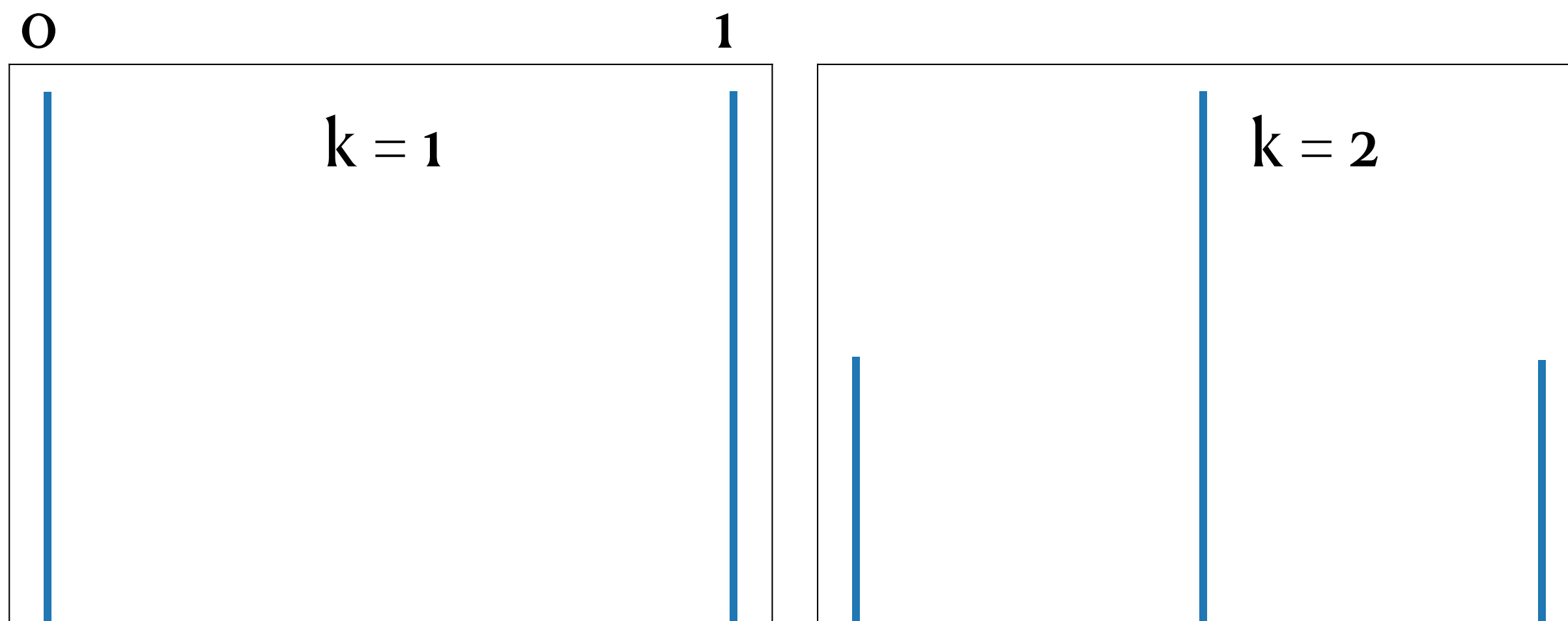
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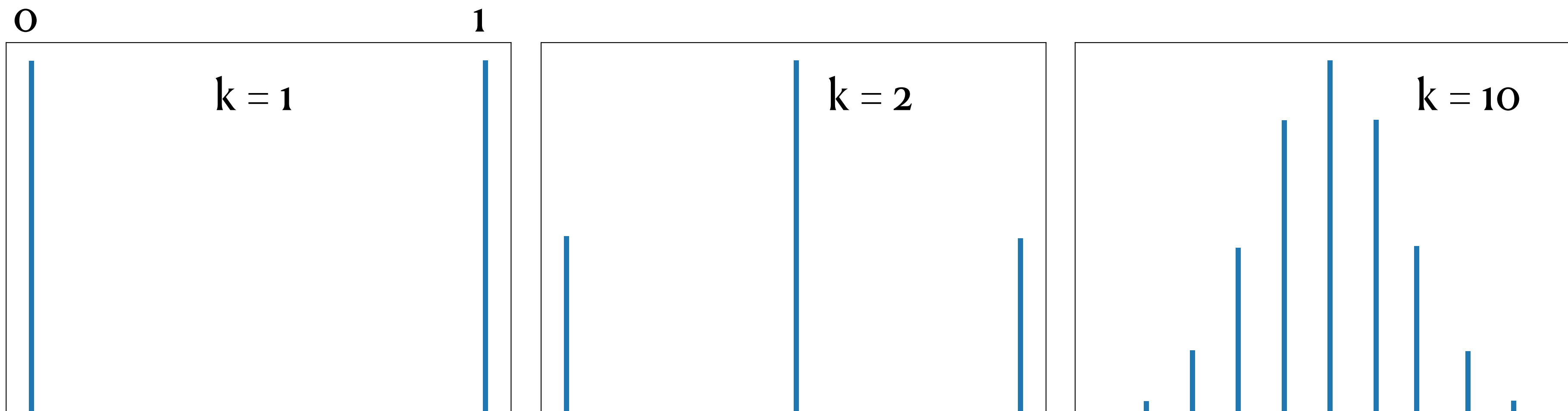
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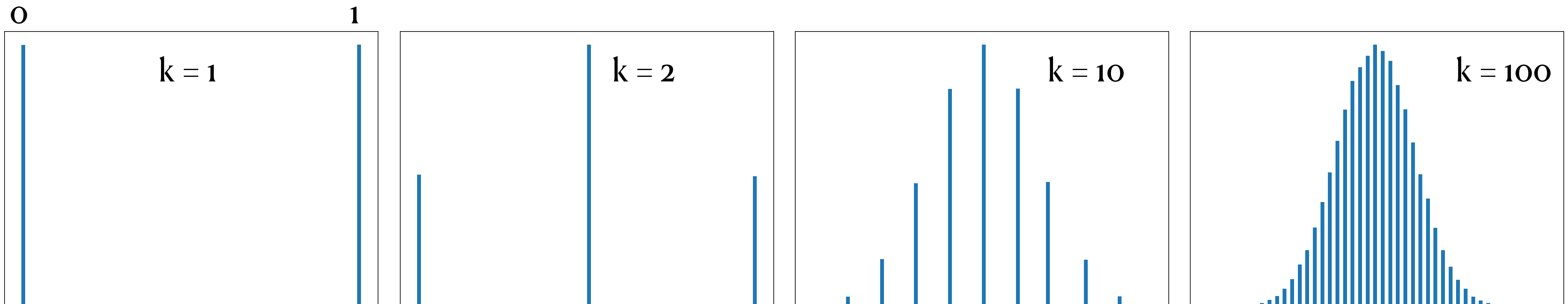
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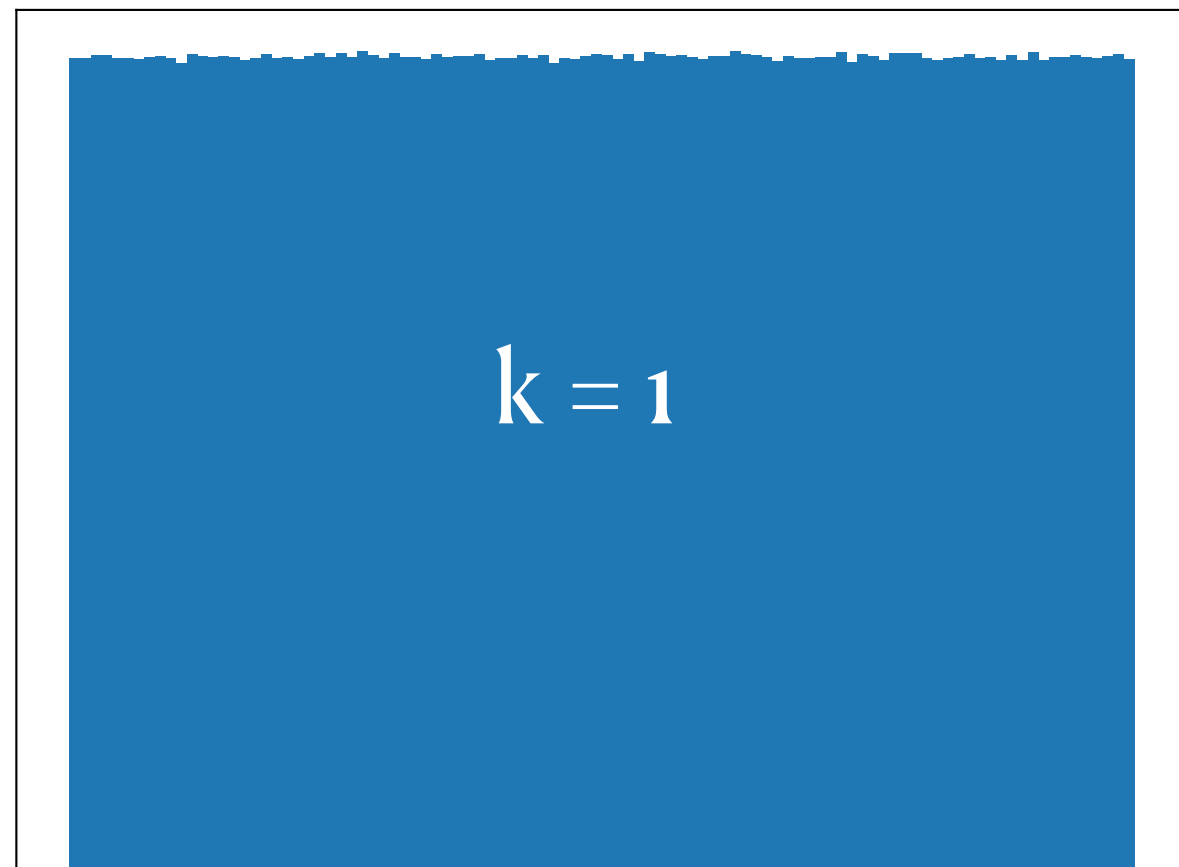
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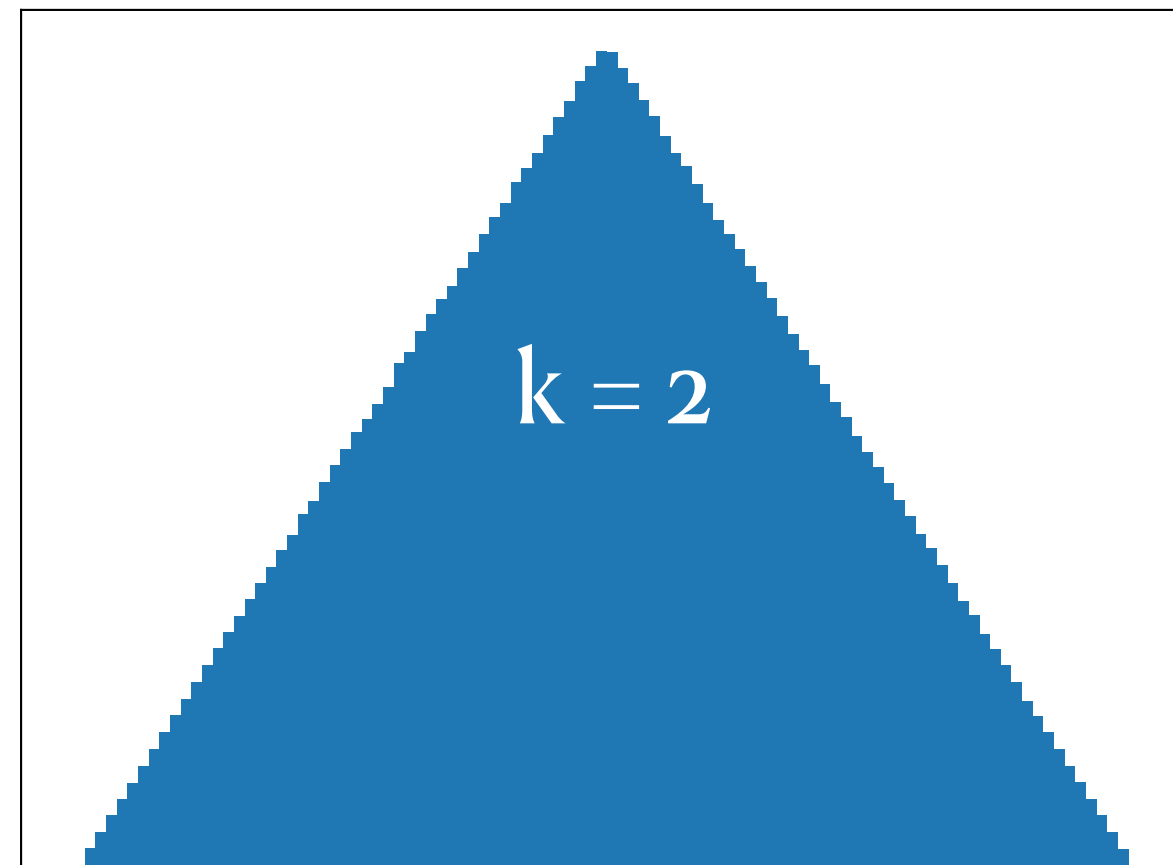
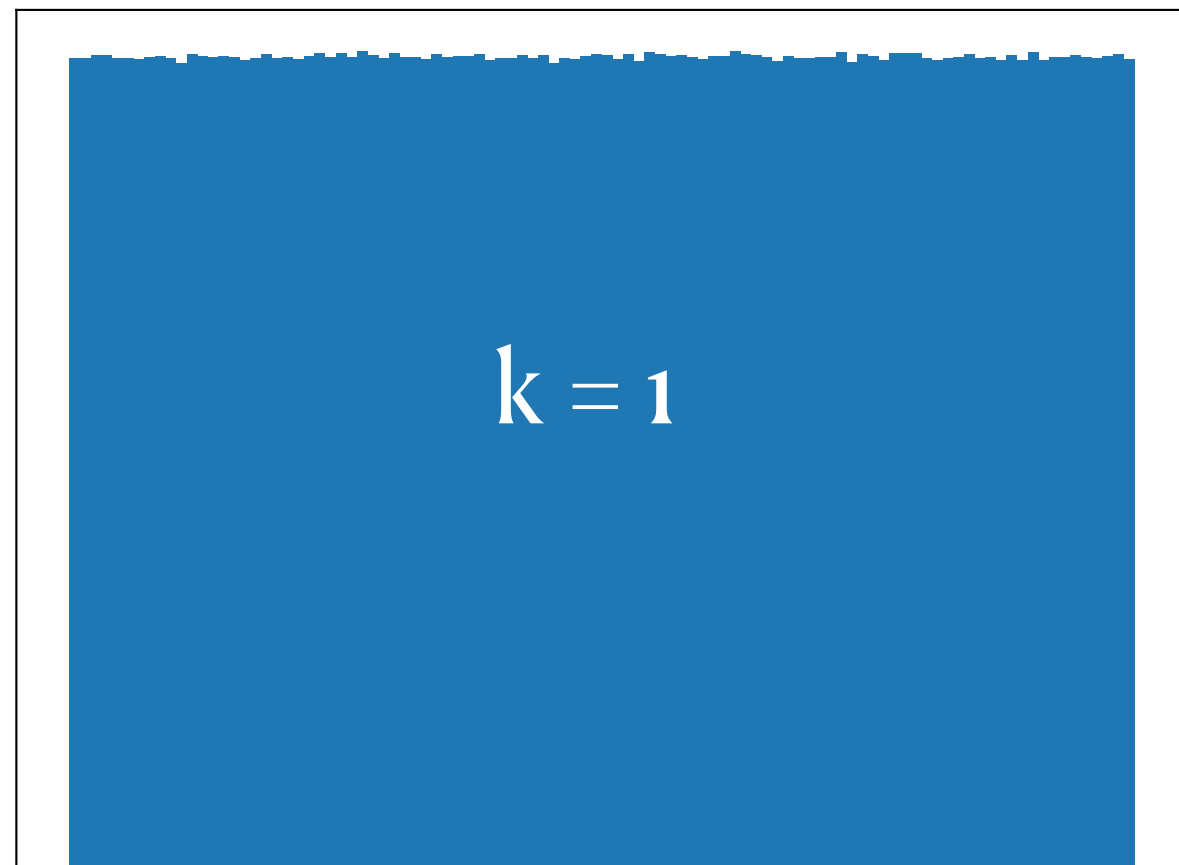
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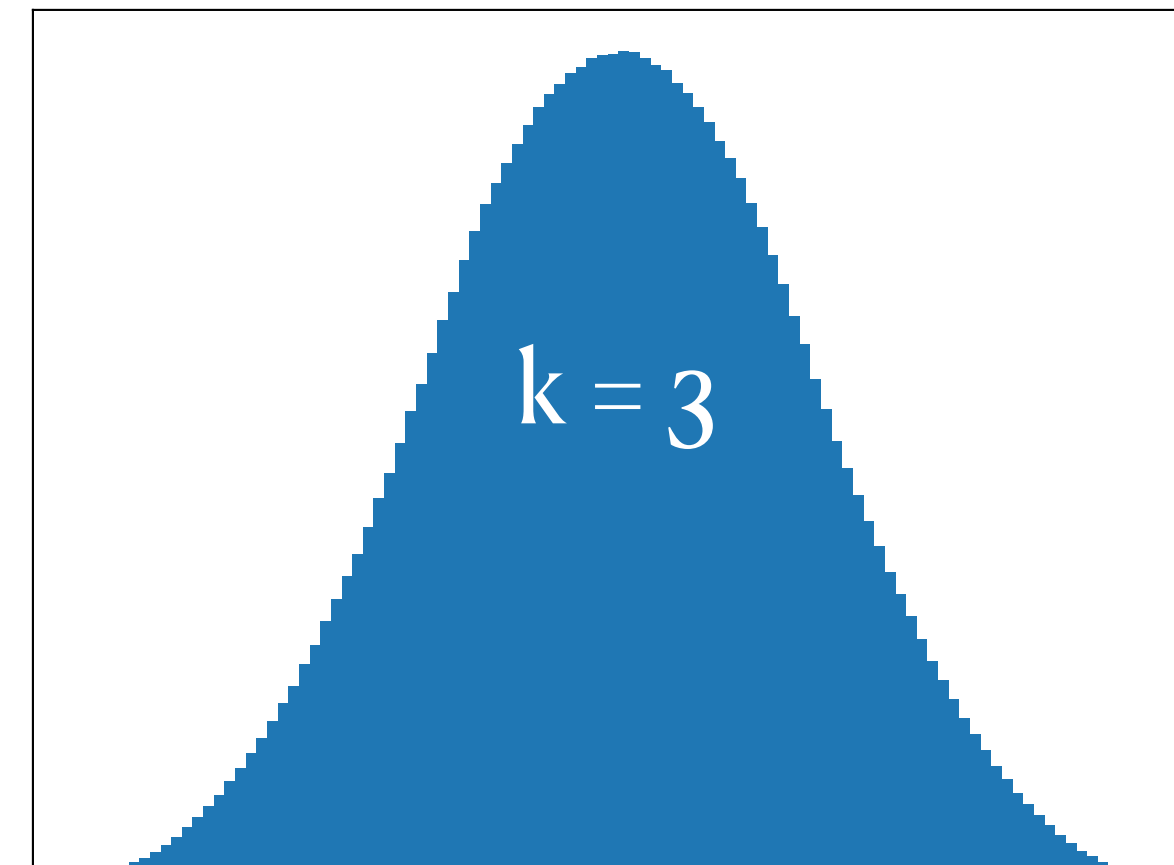
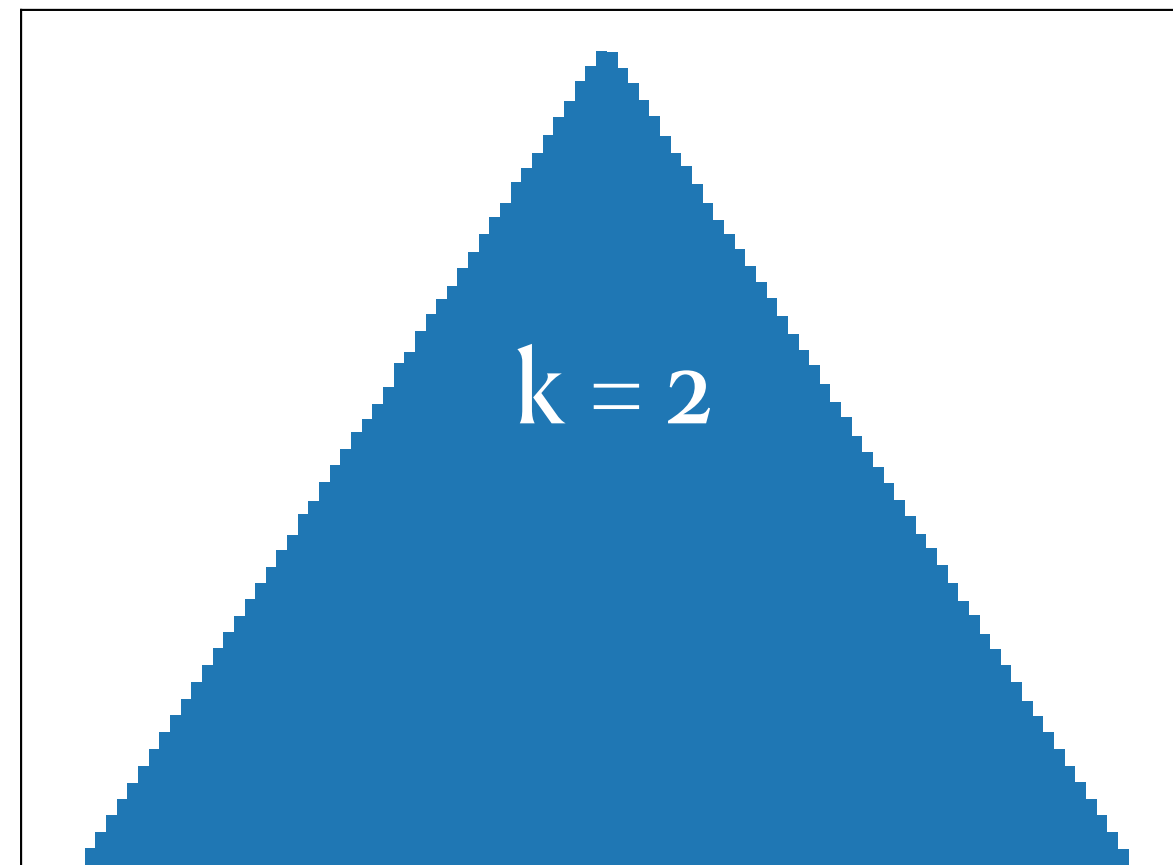
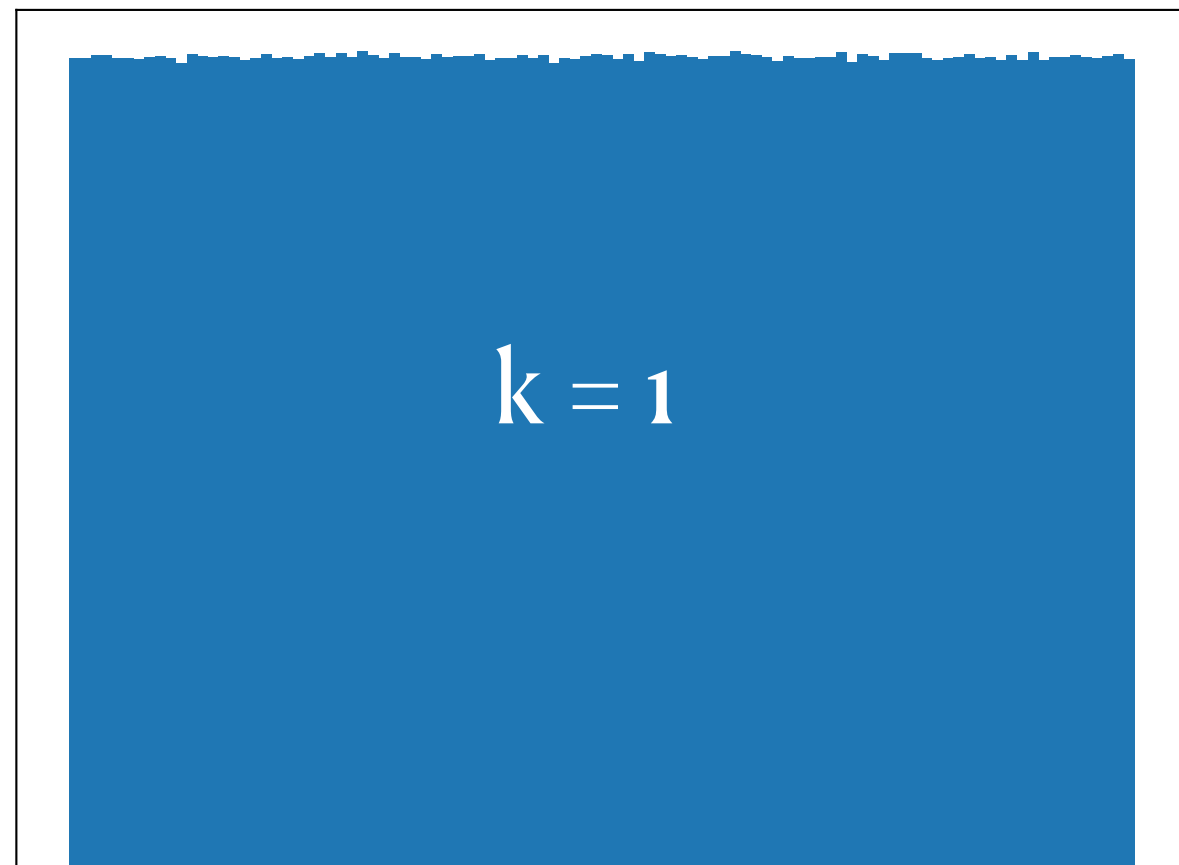
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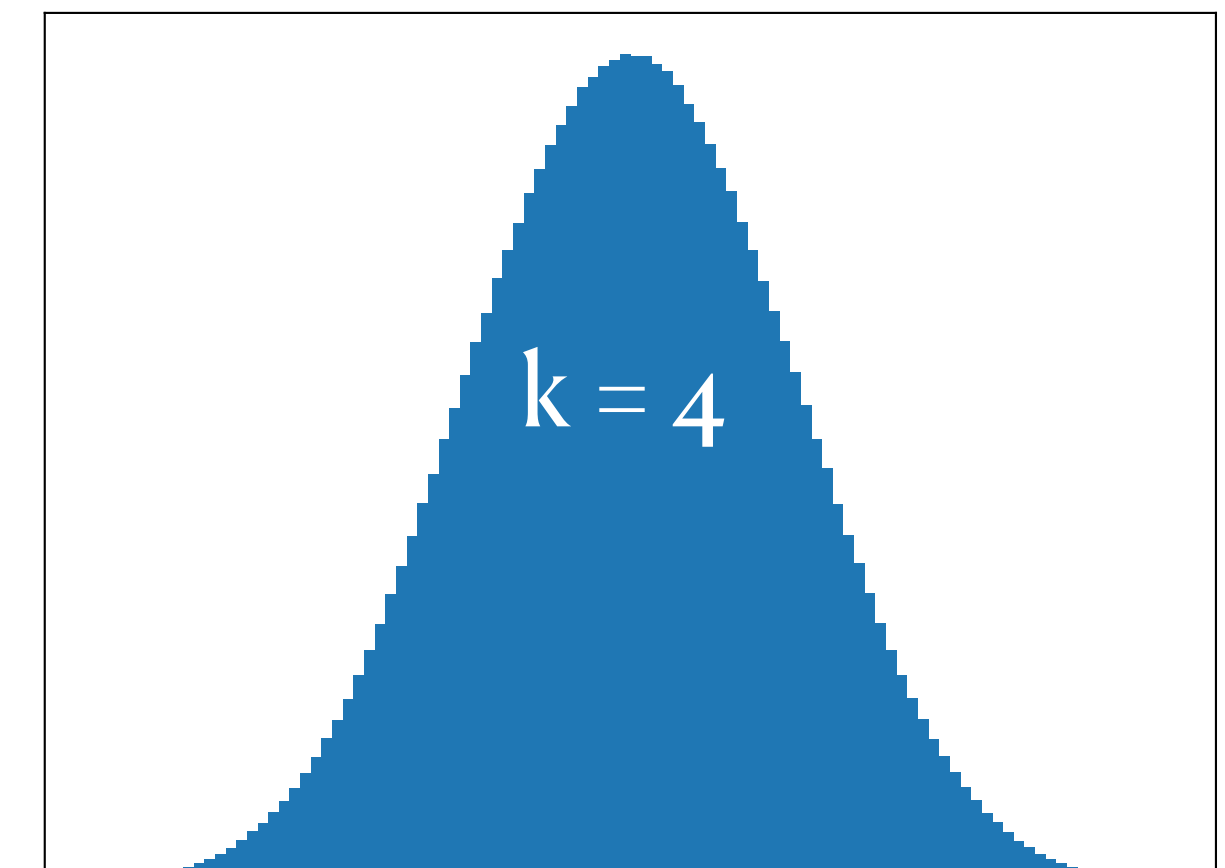
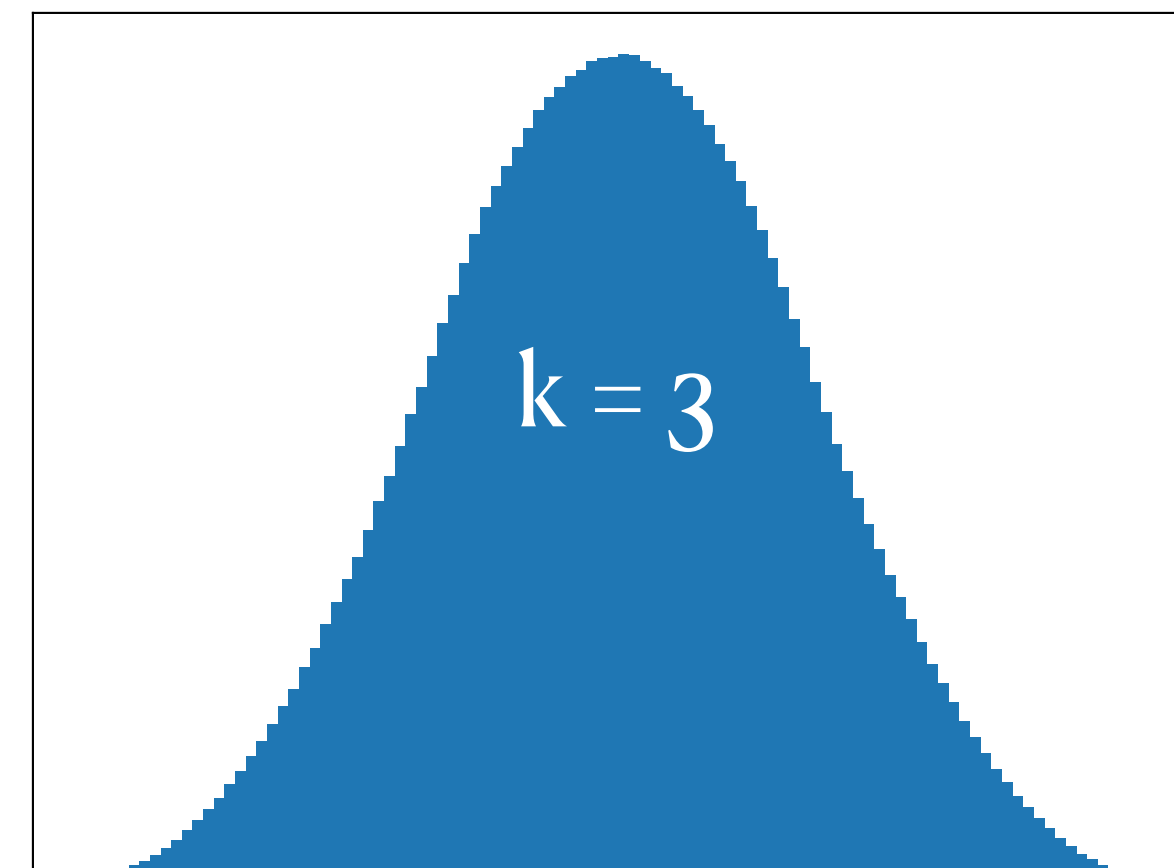
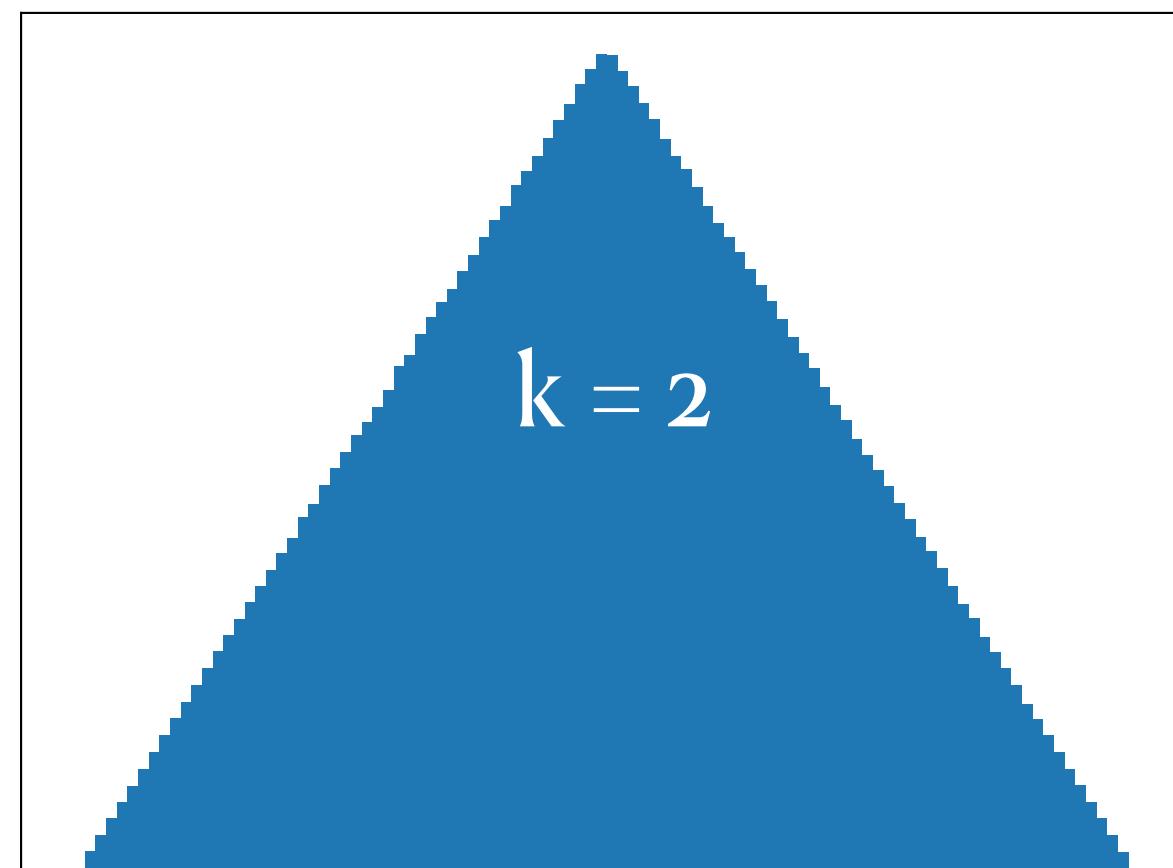
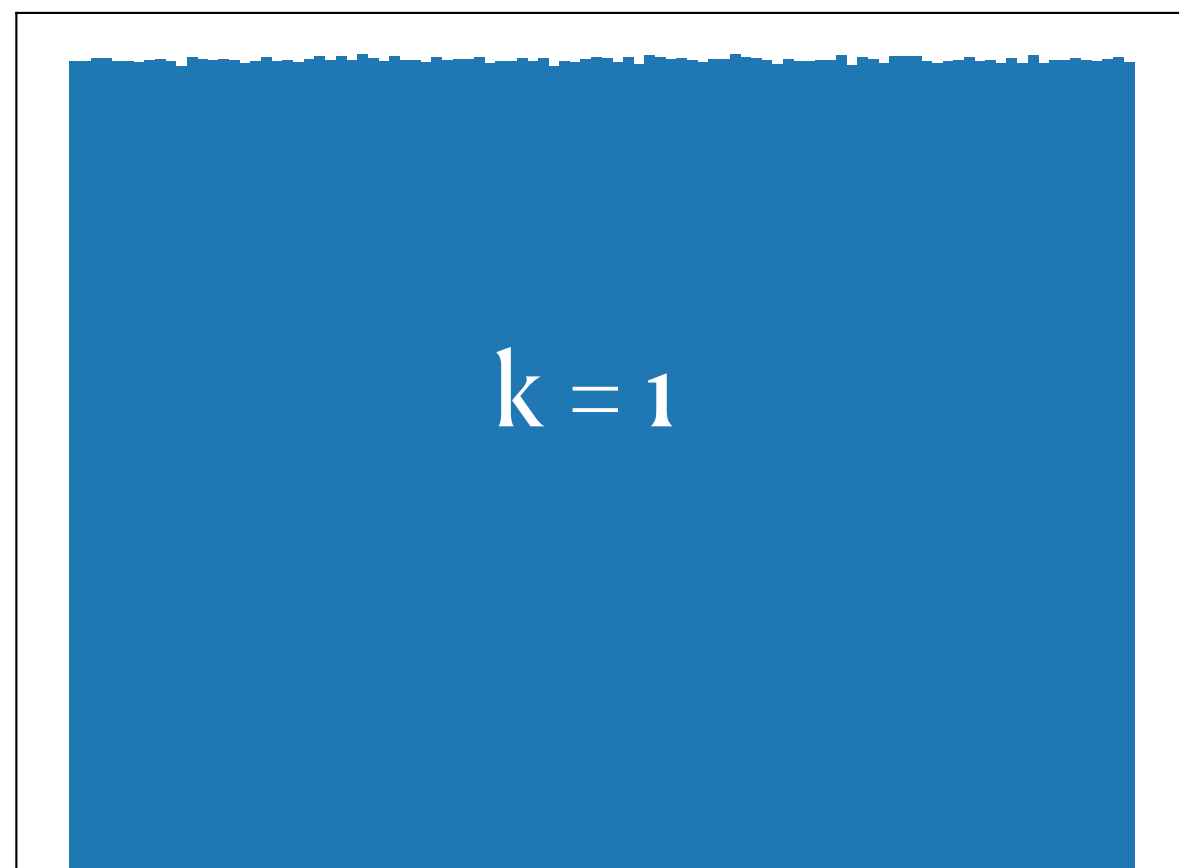
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X_1, X_2, \dots

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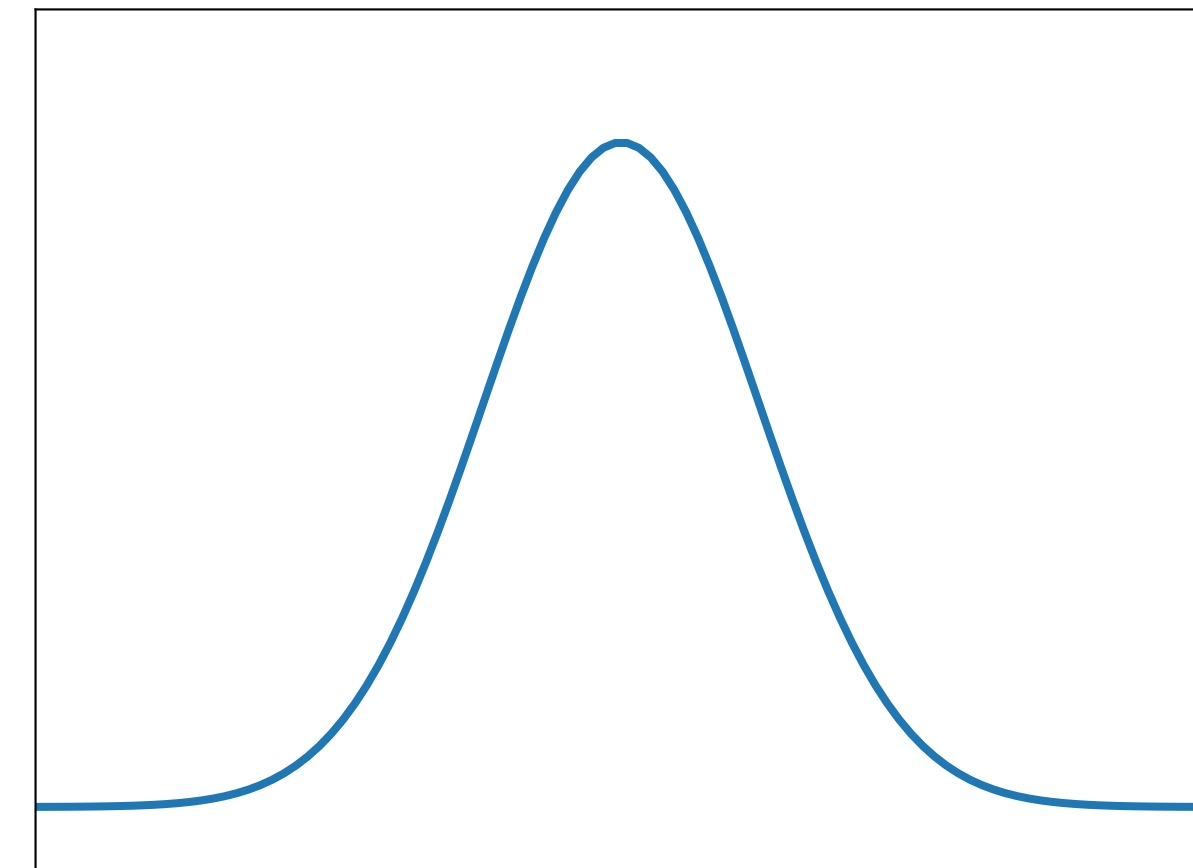
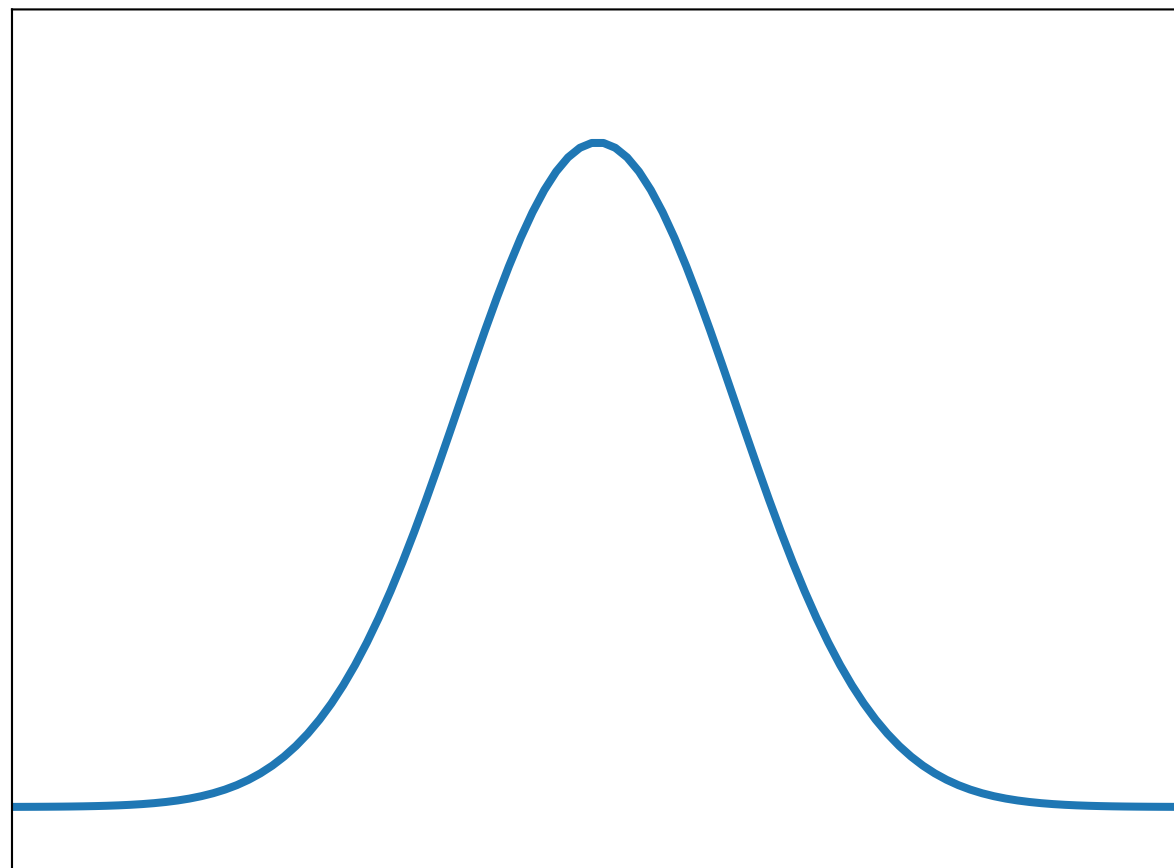
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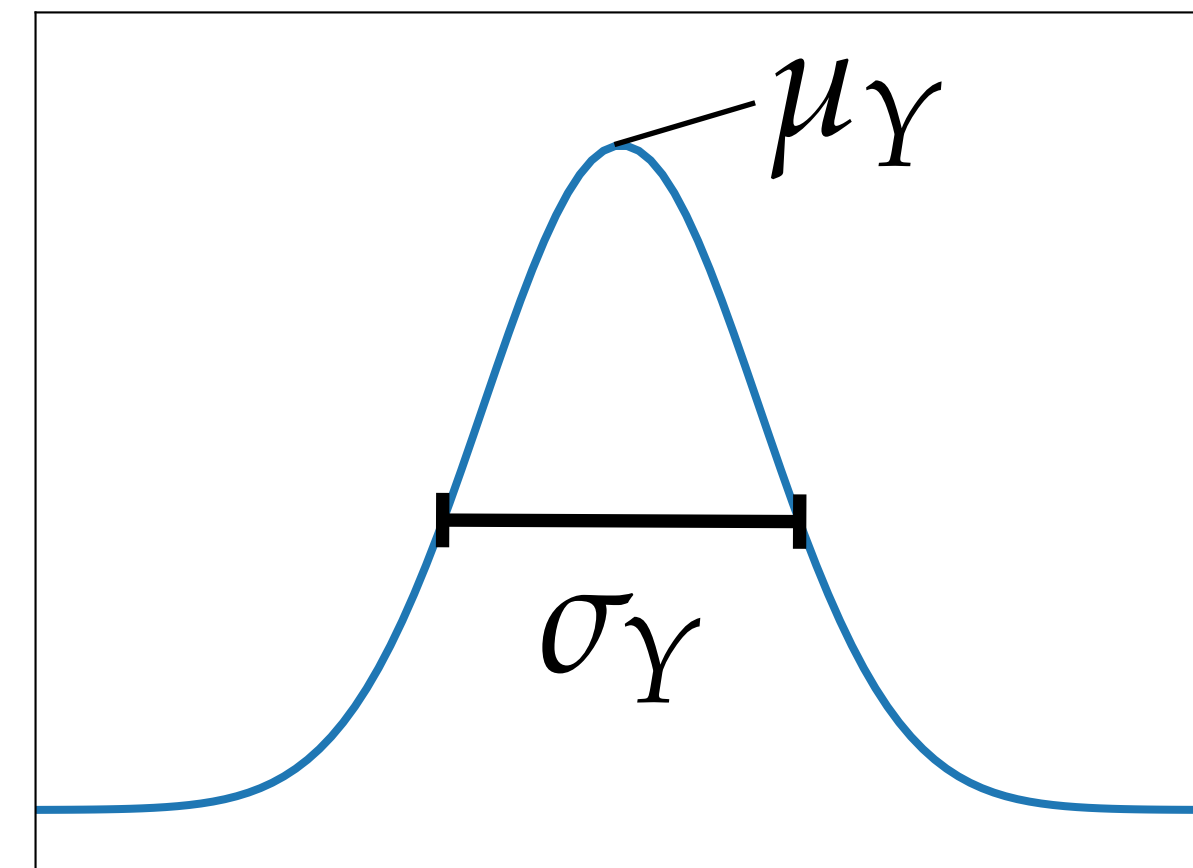
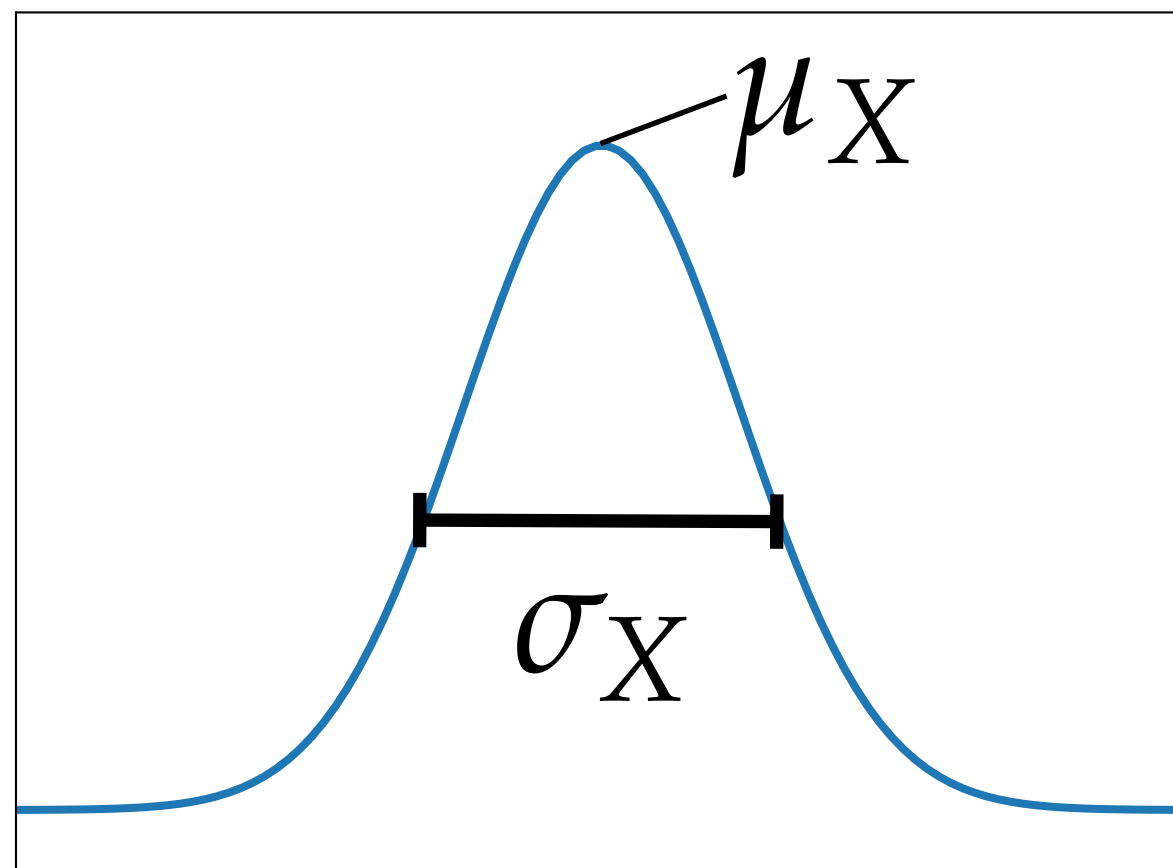
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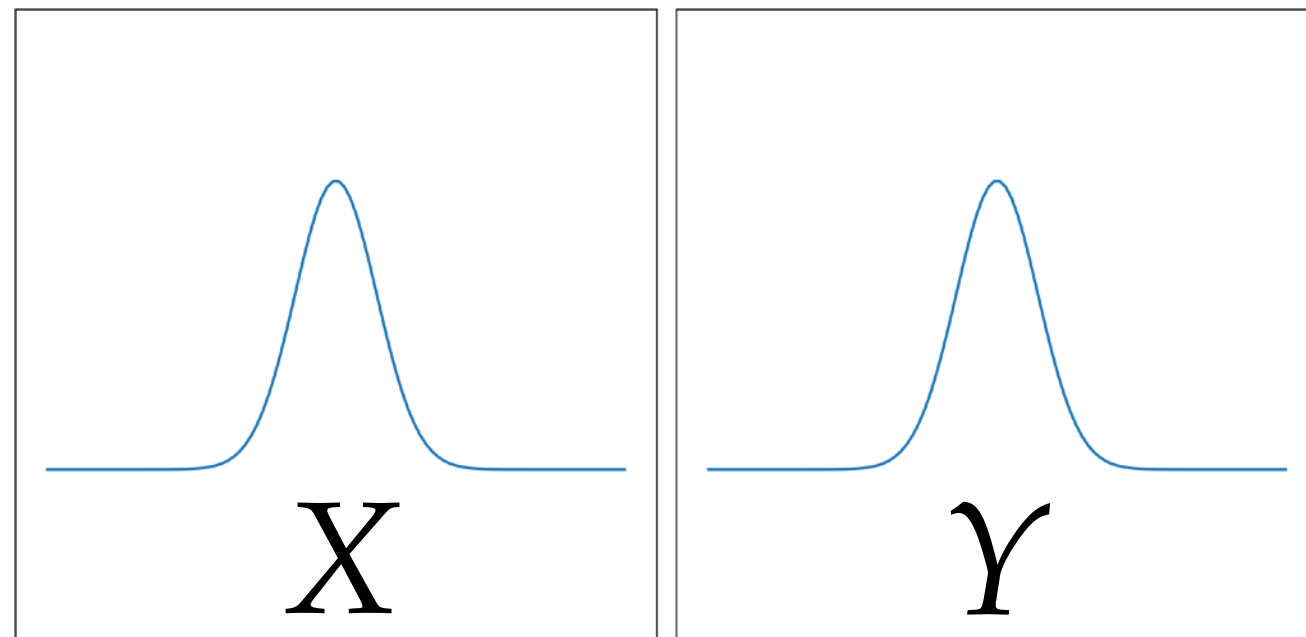
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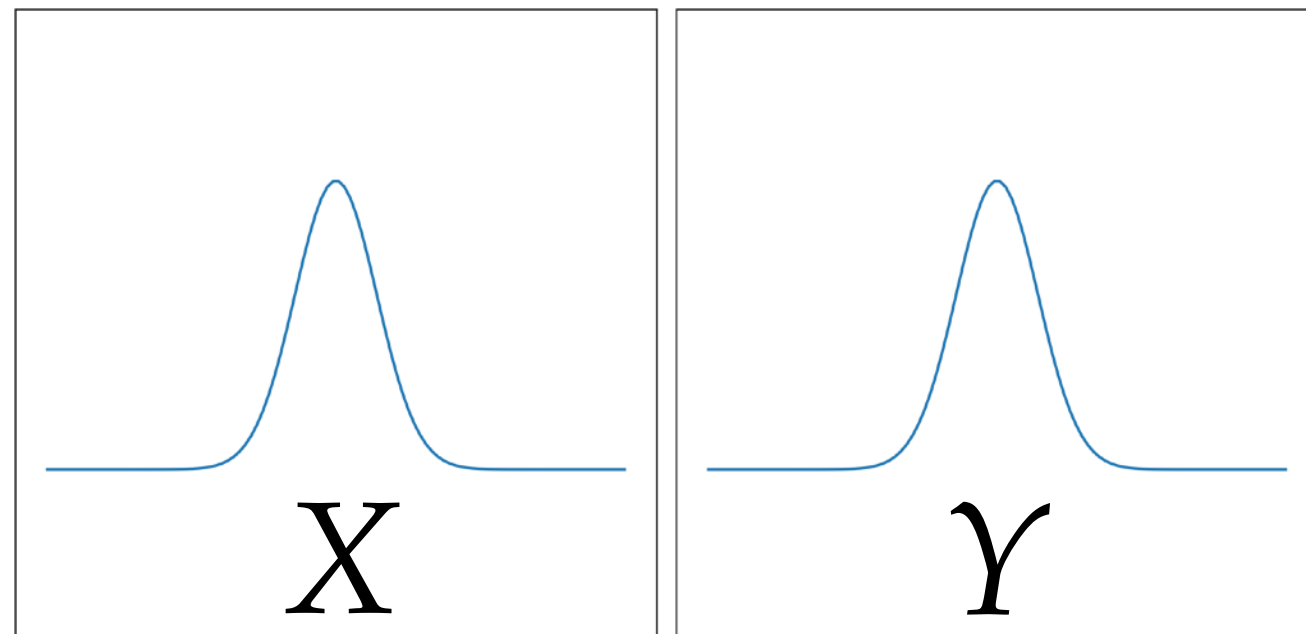
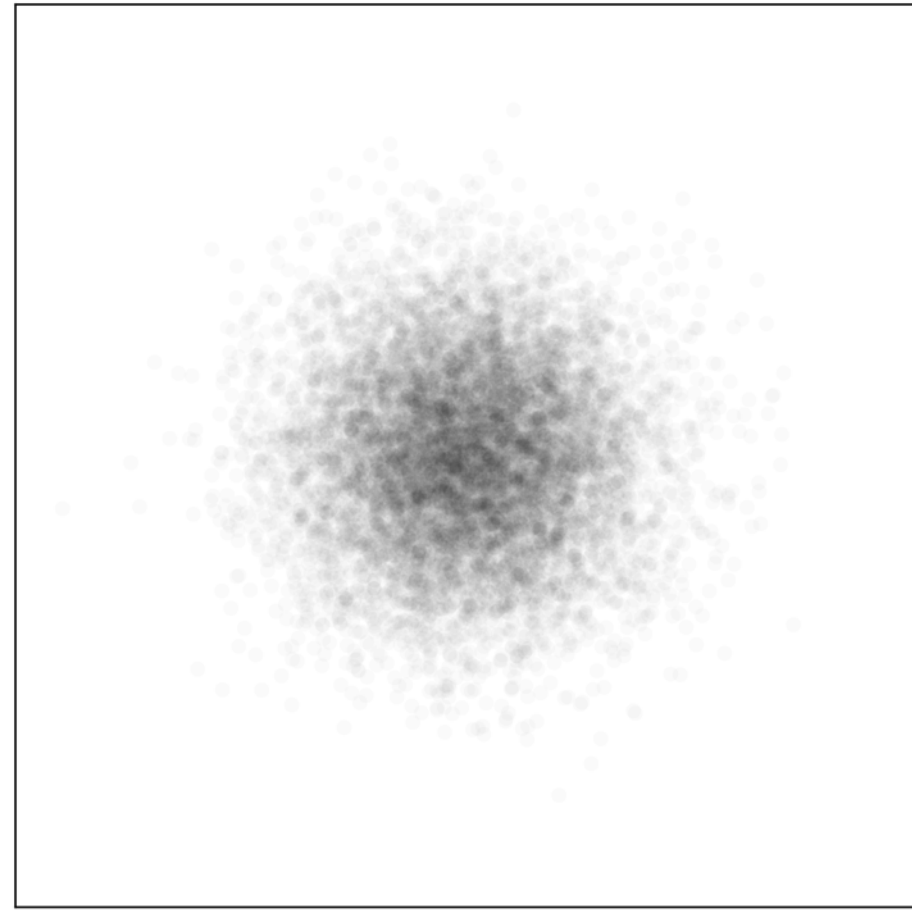


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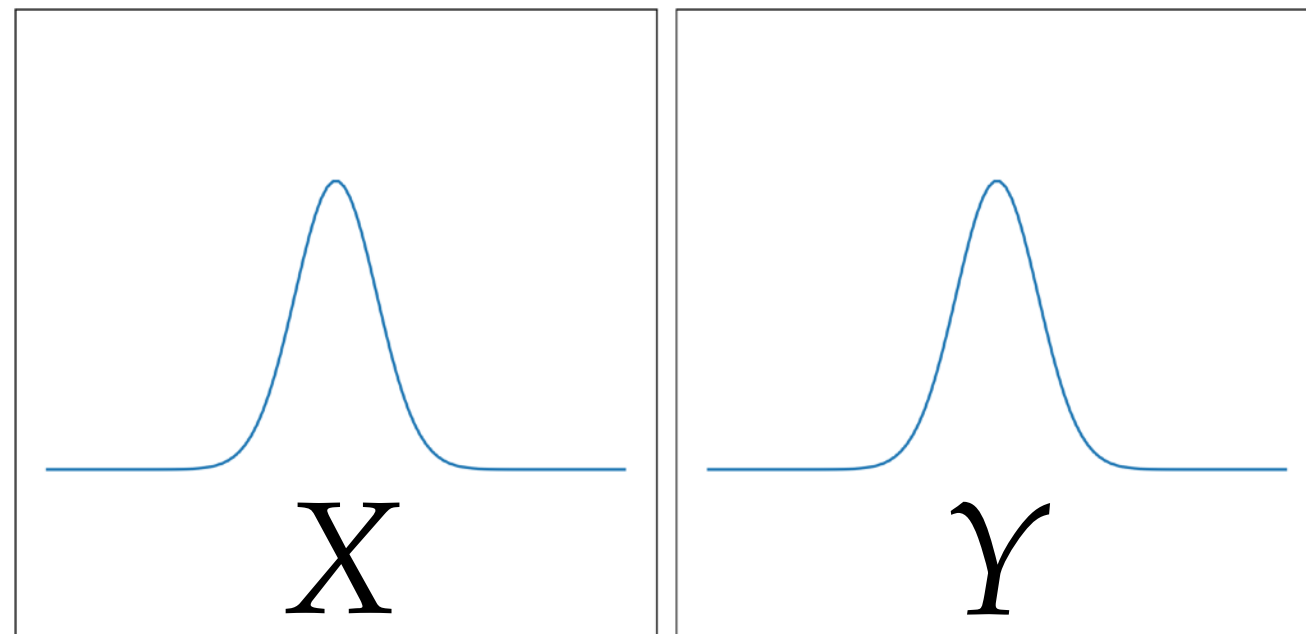
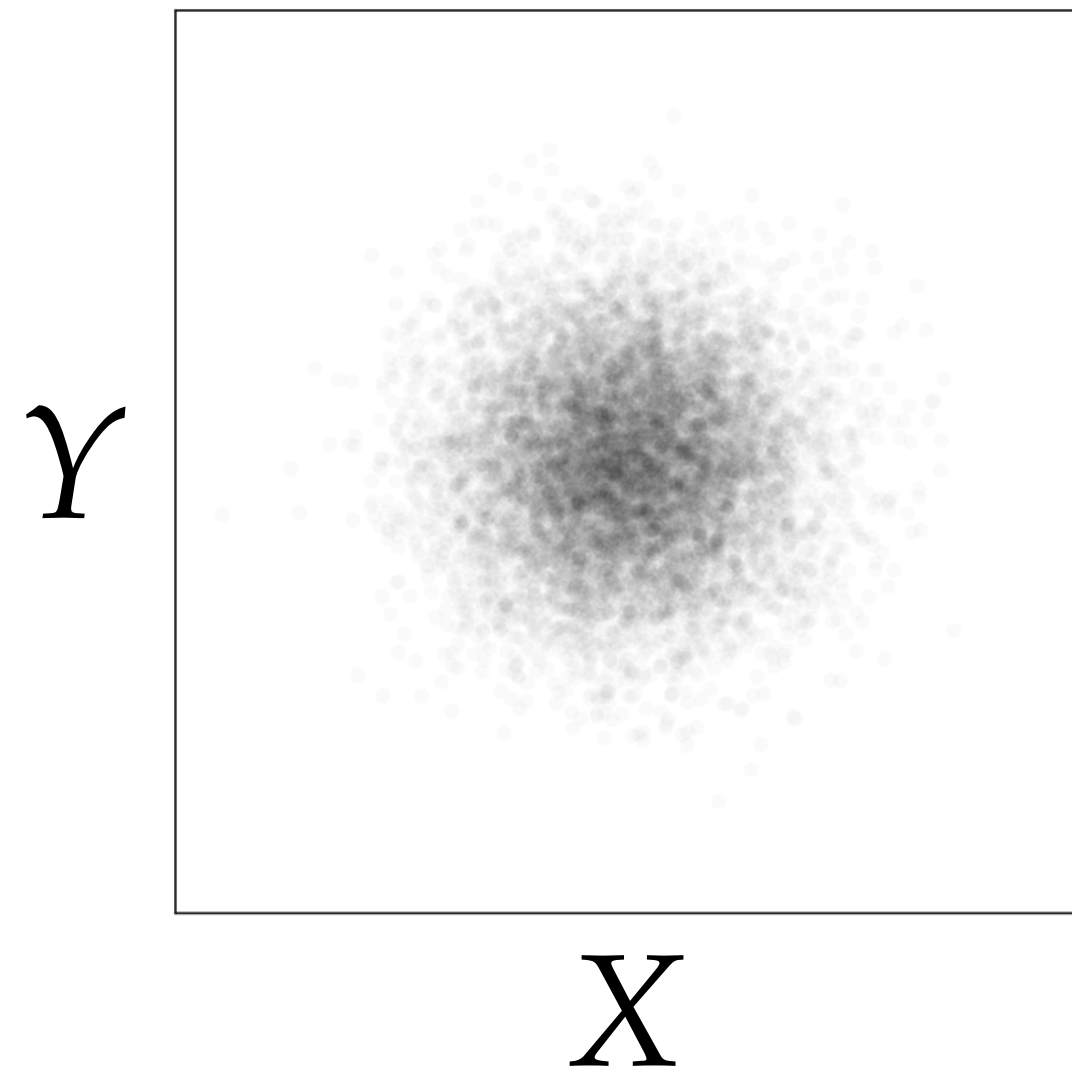
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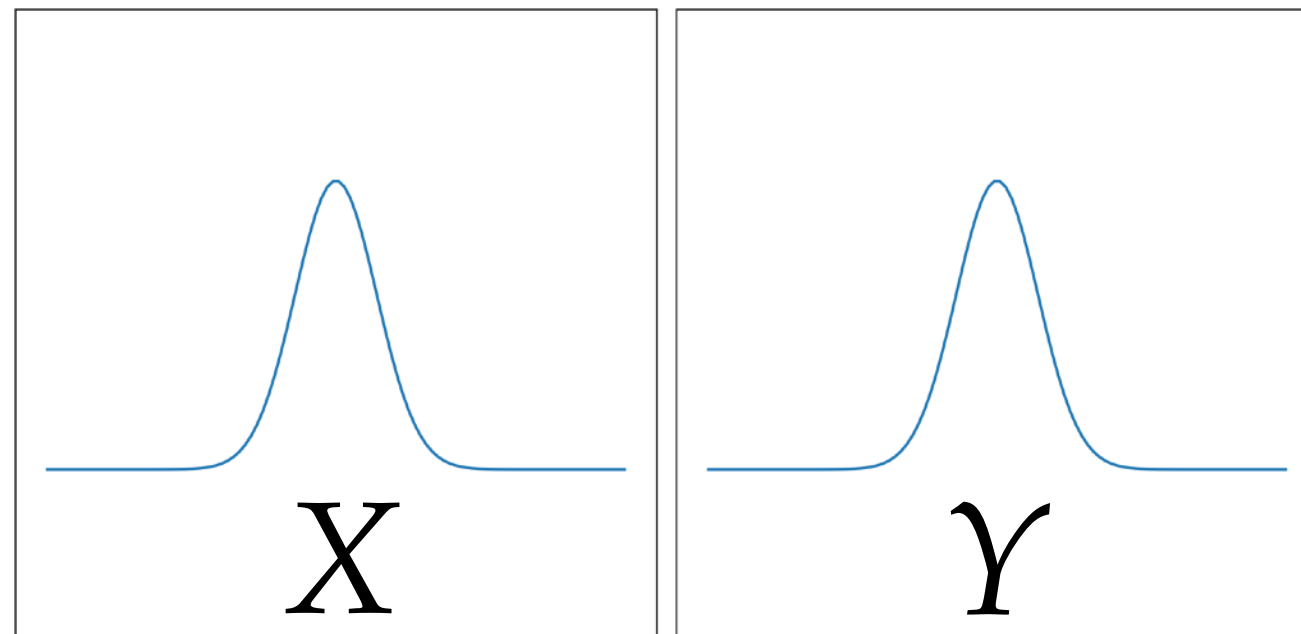
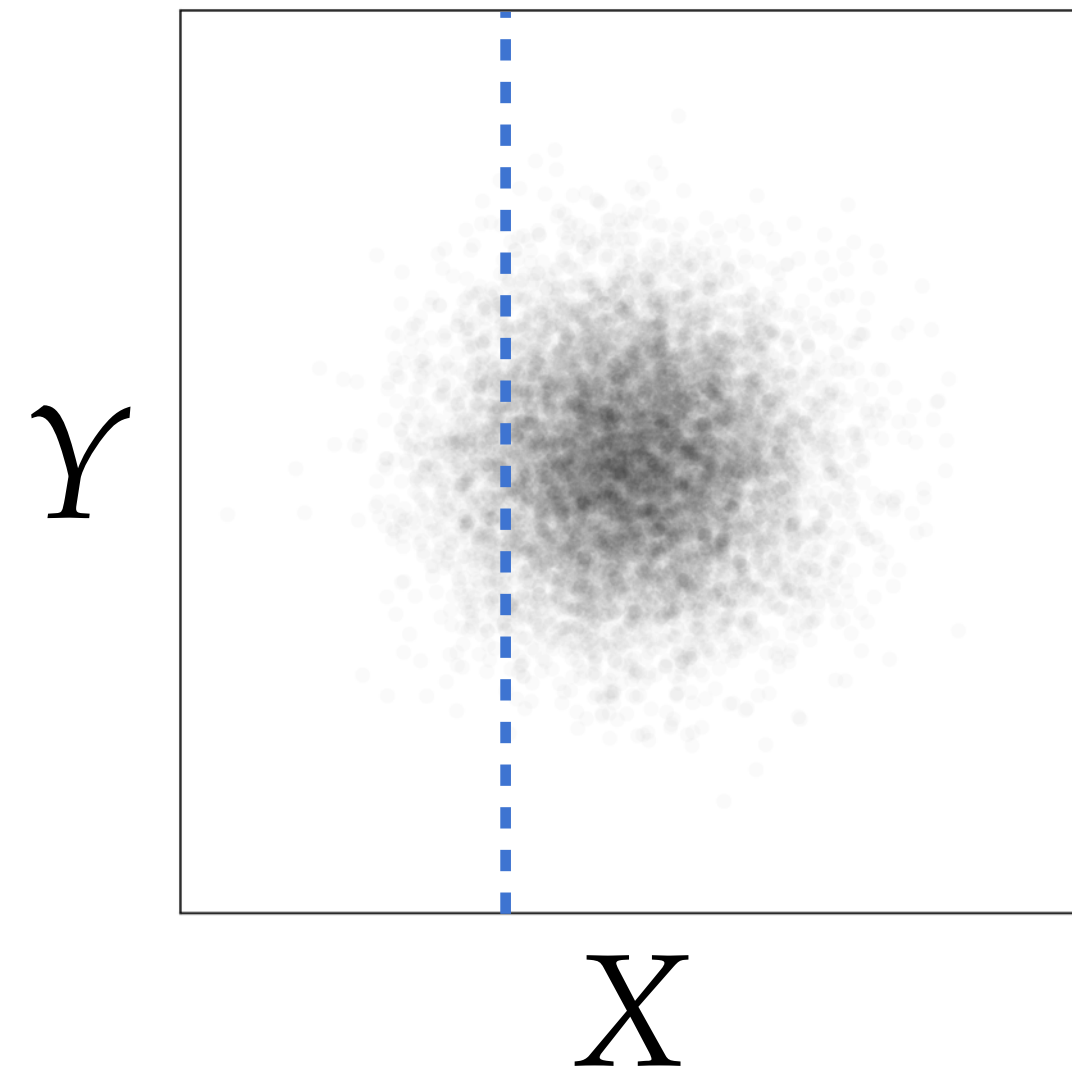
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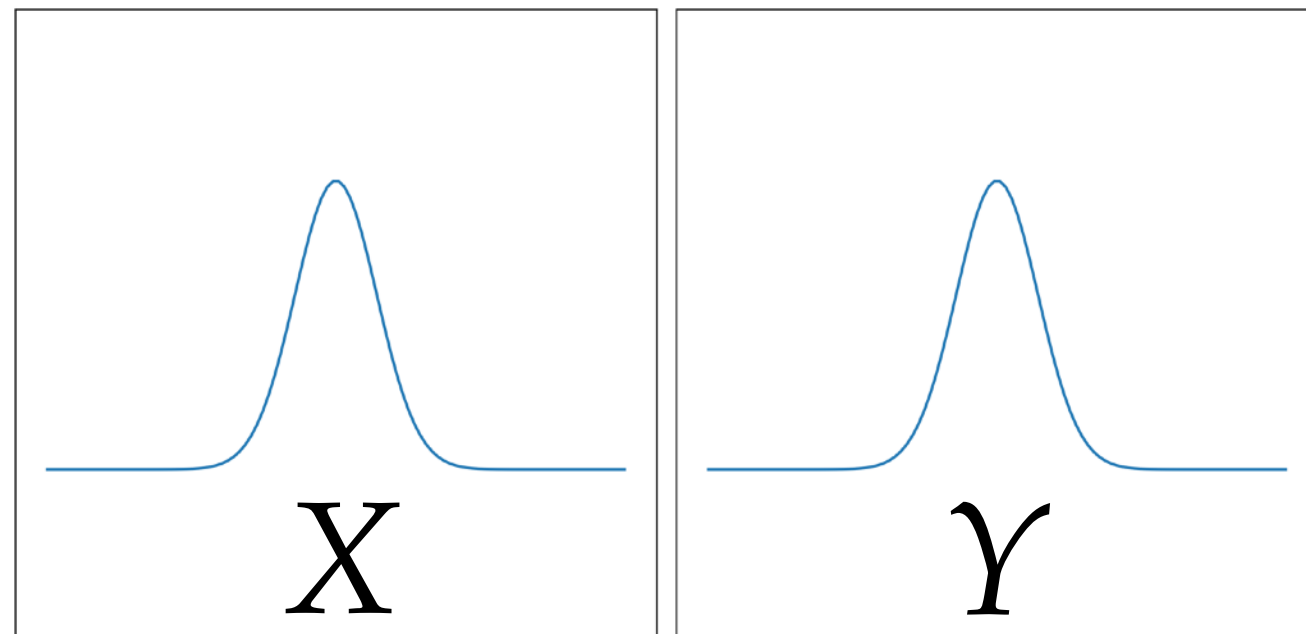
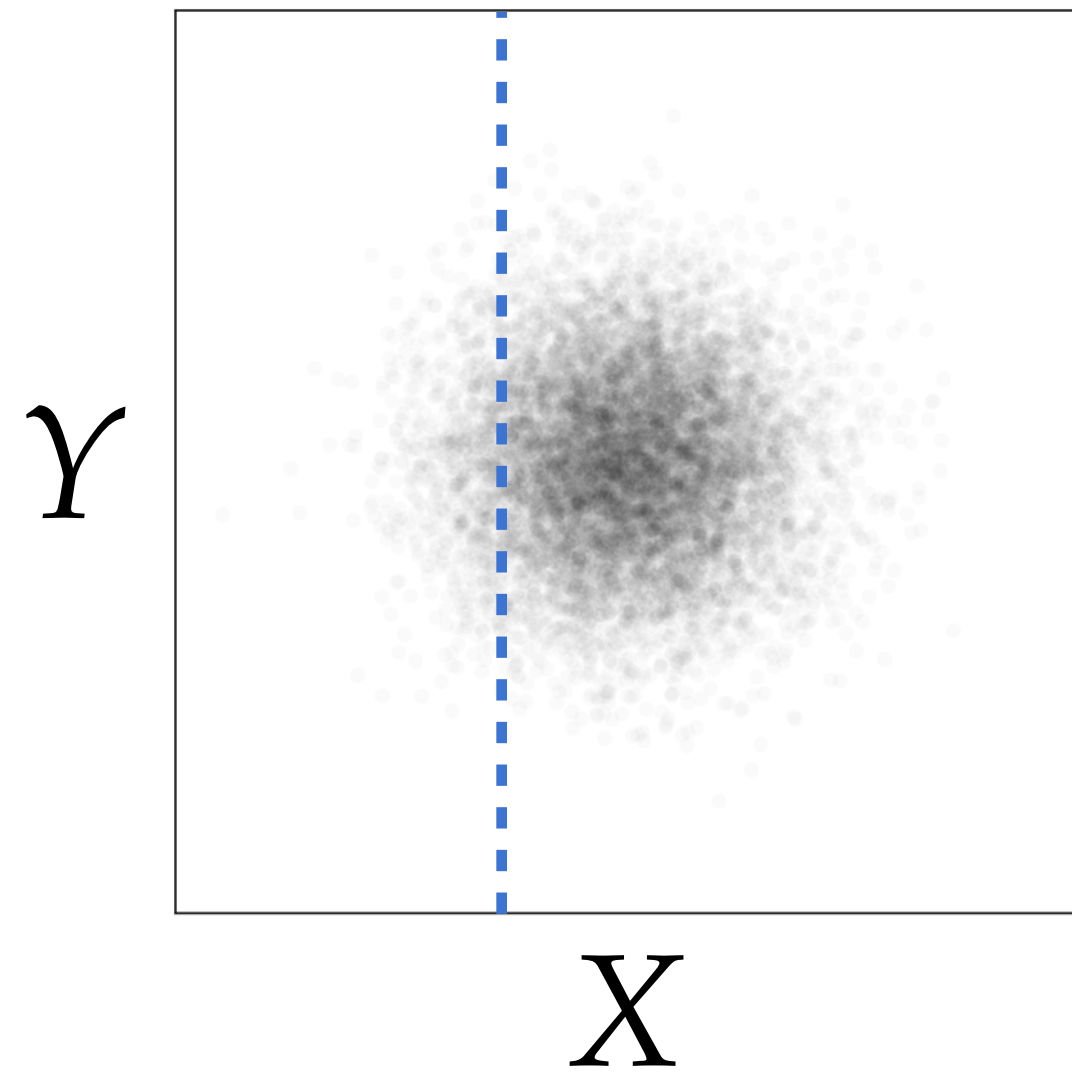
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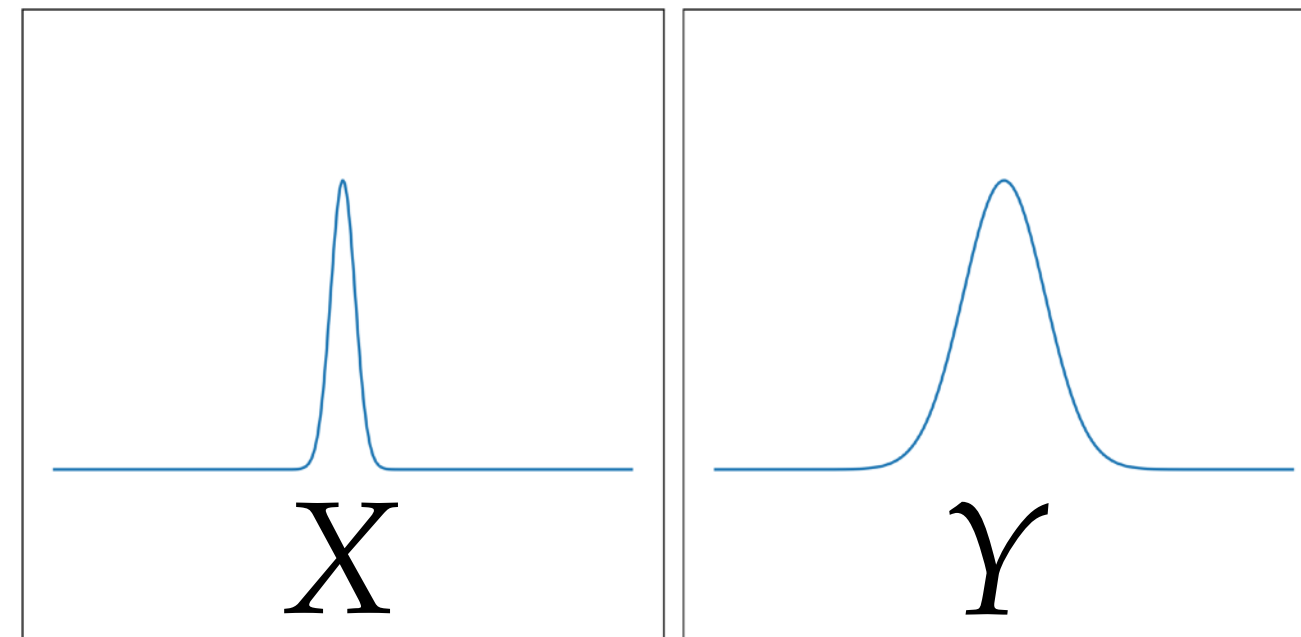
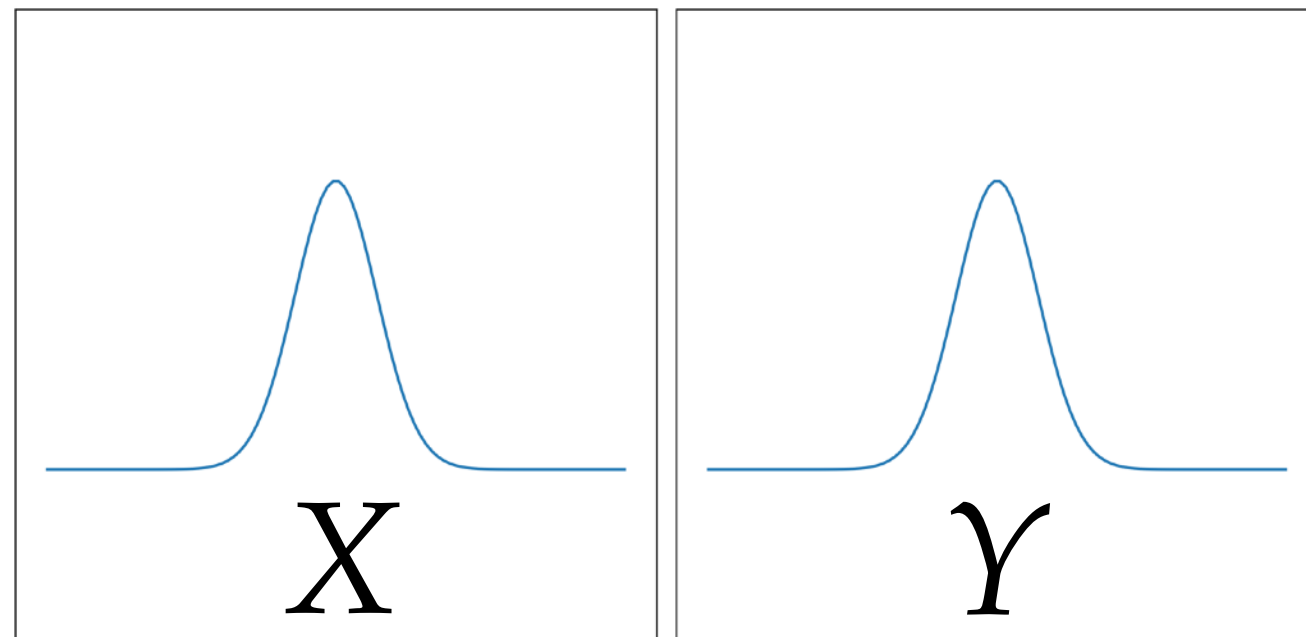
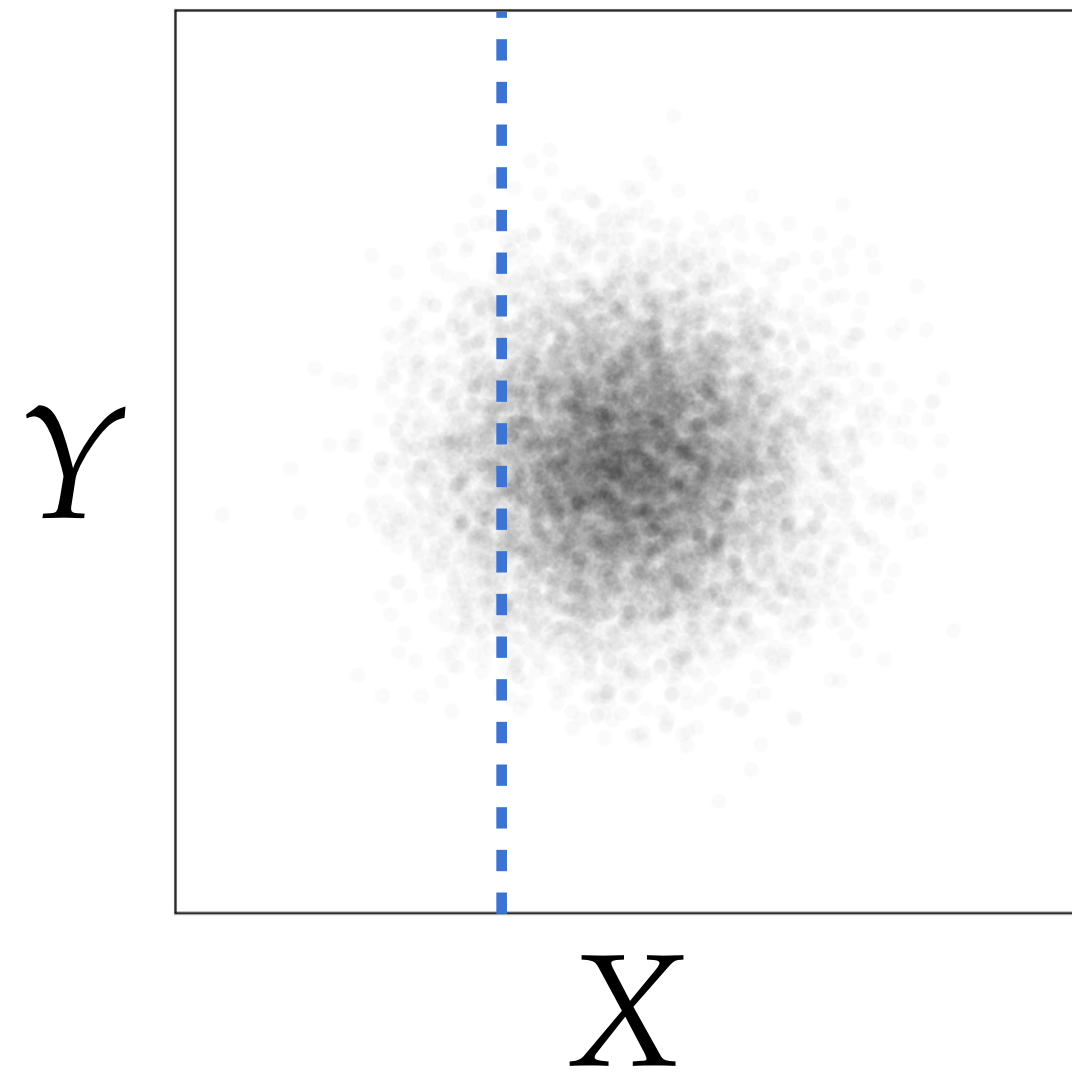
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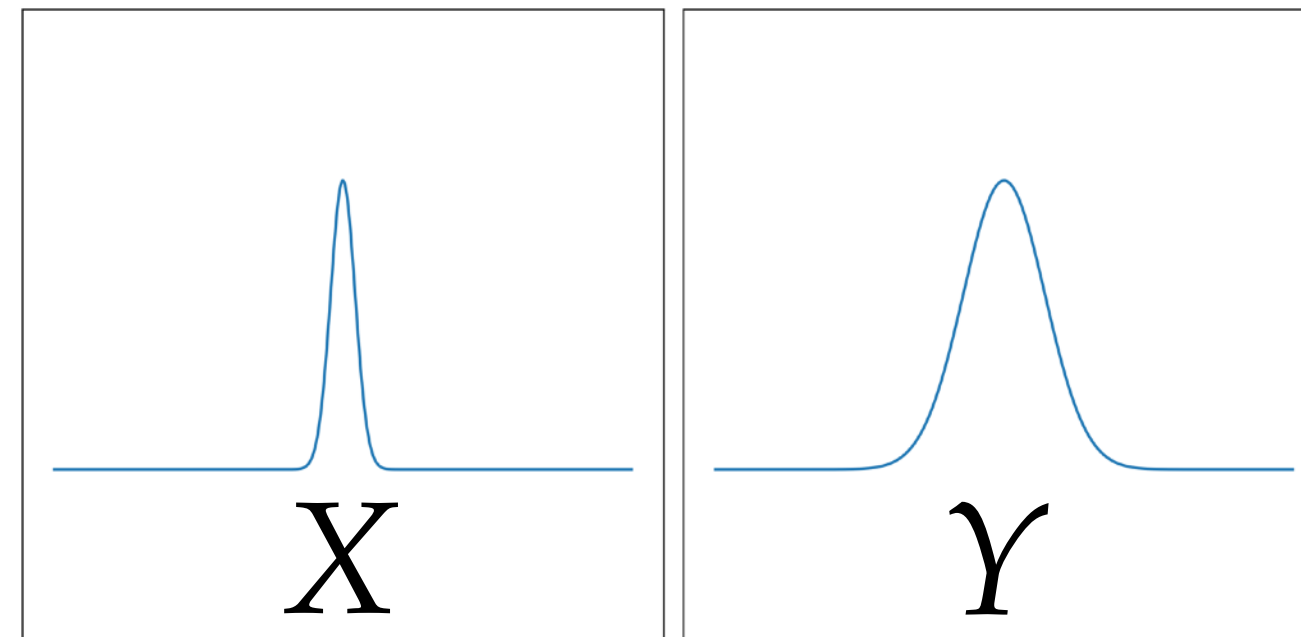
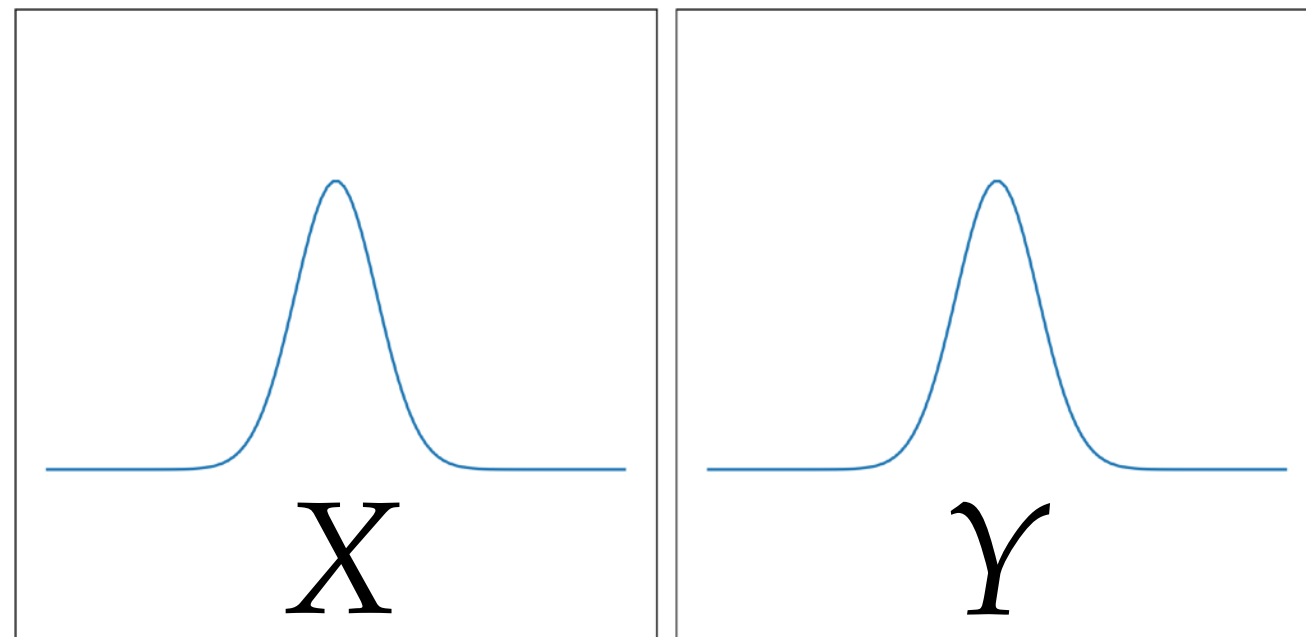
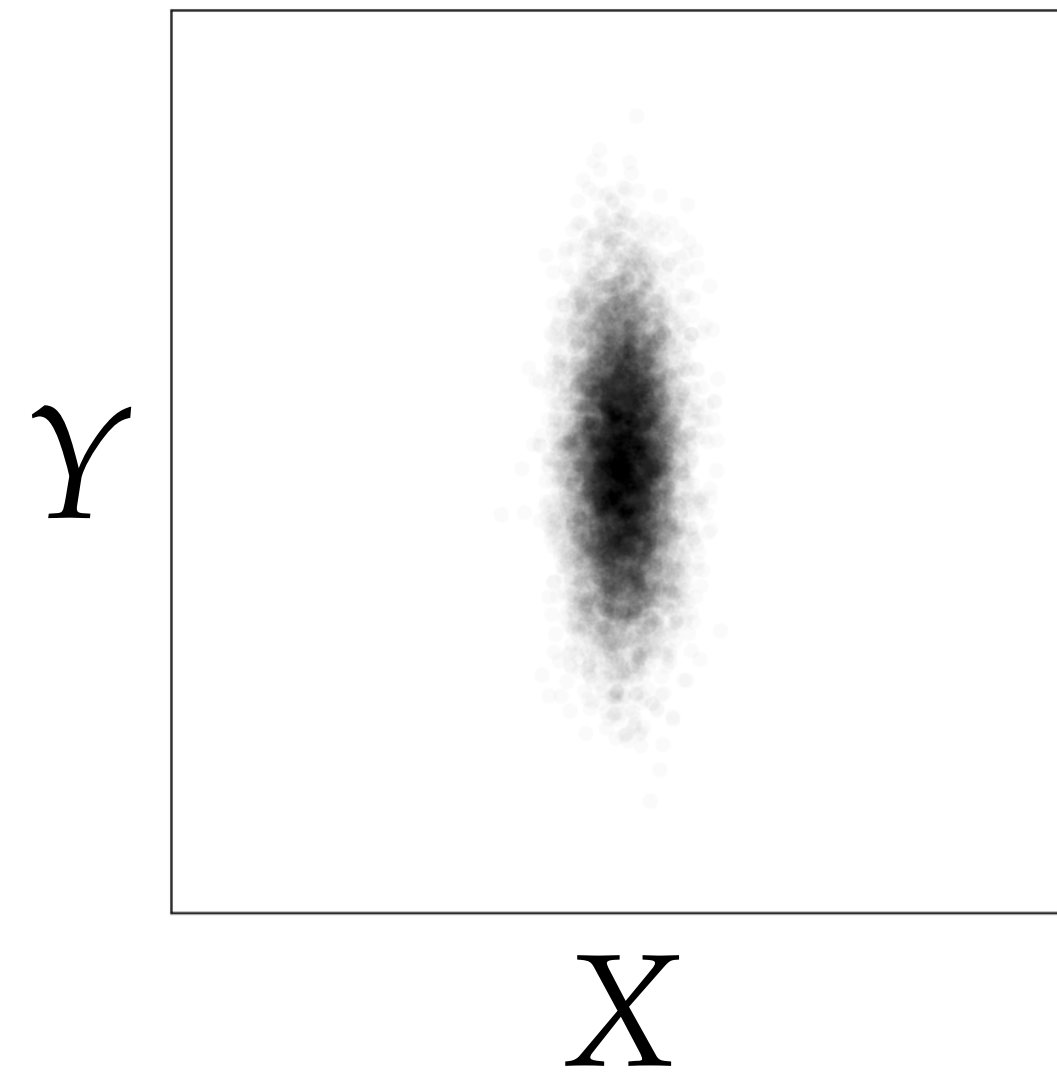
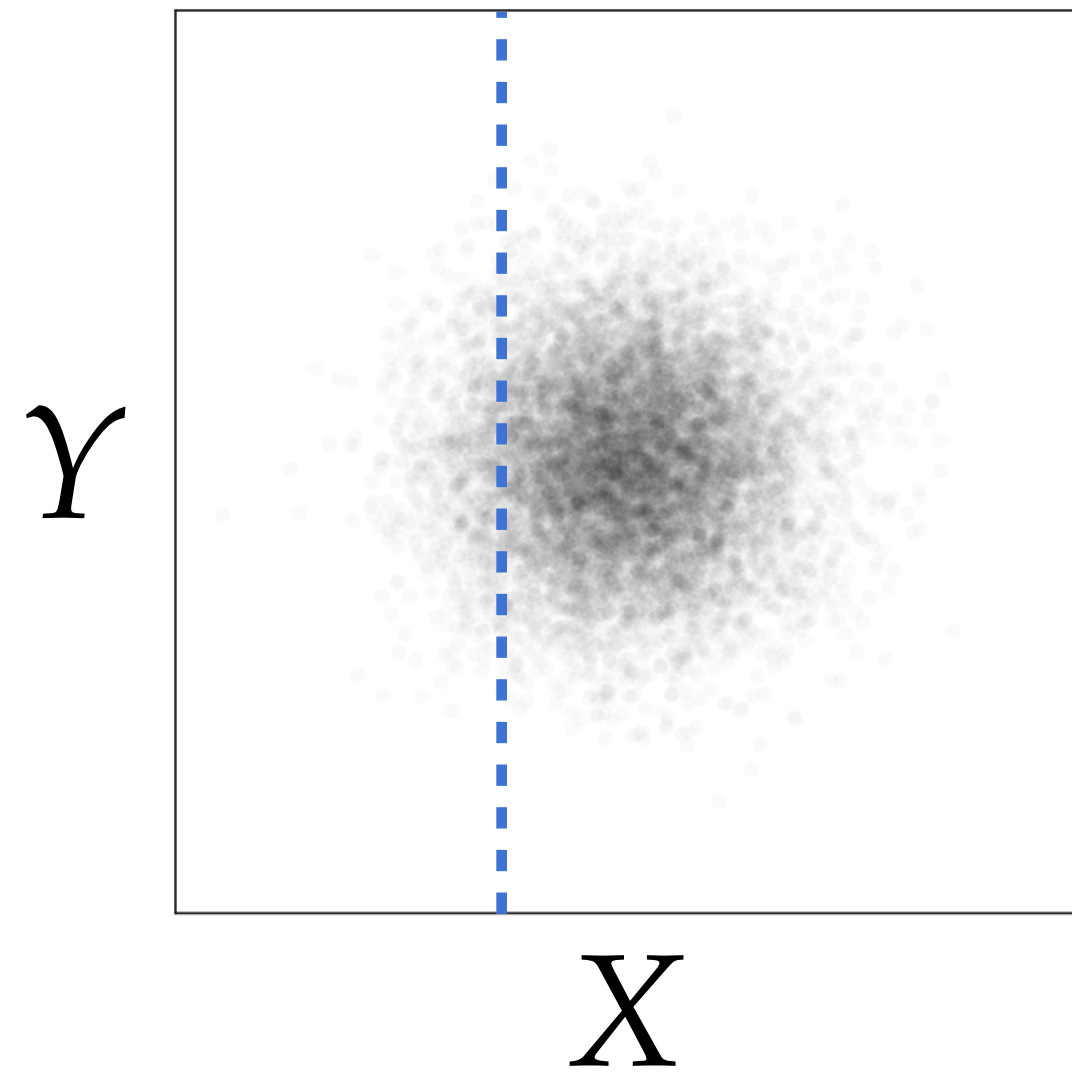
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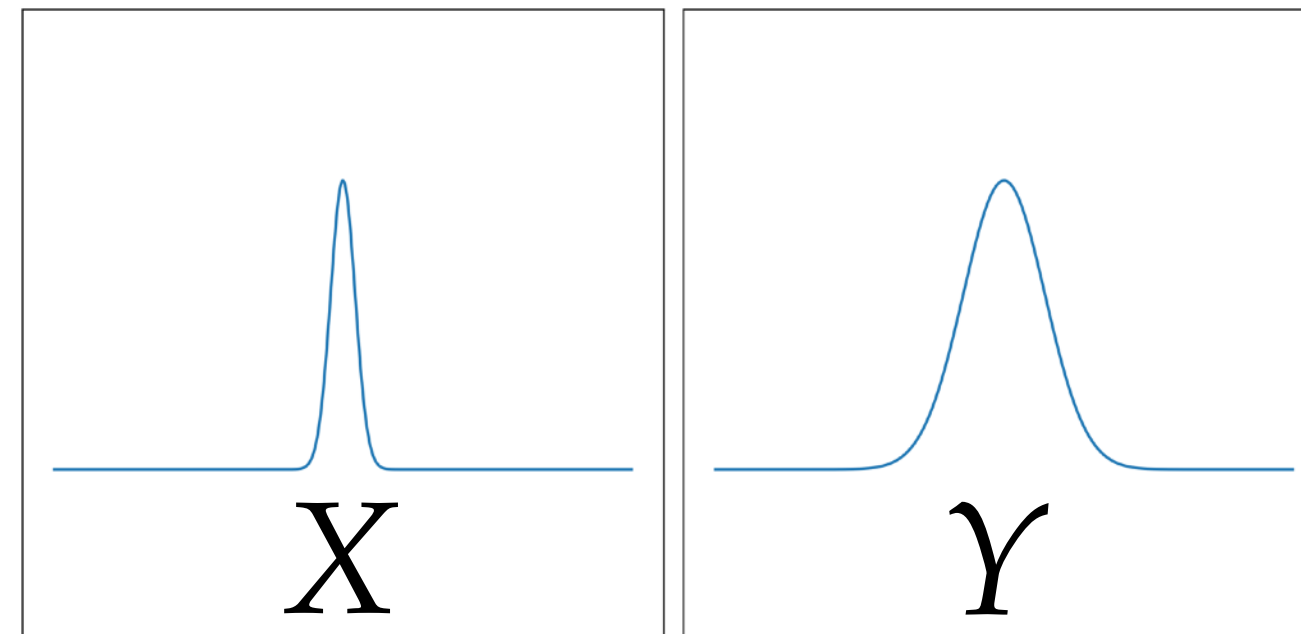
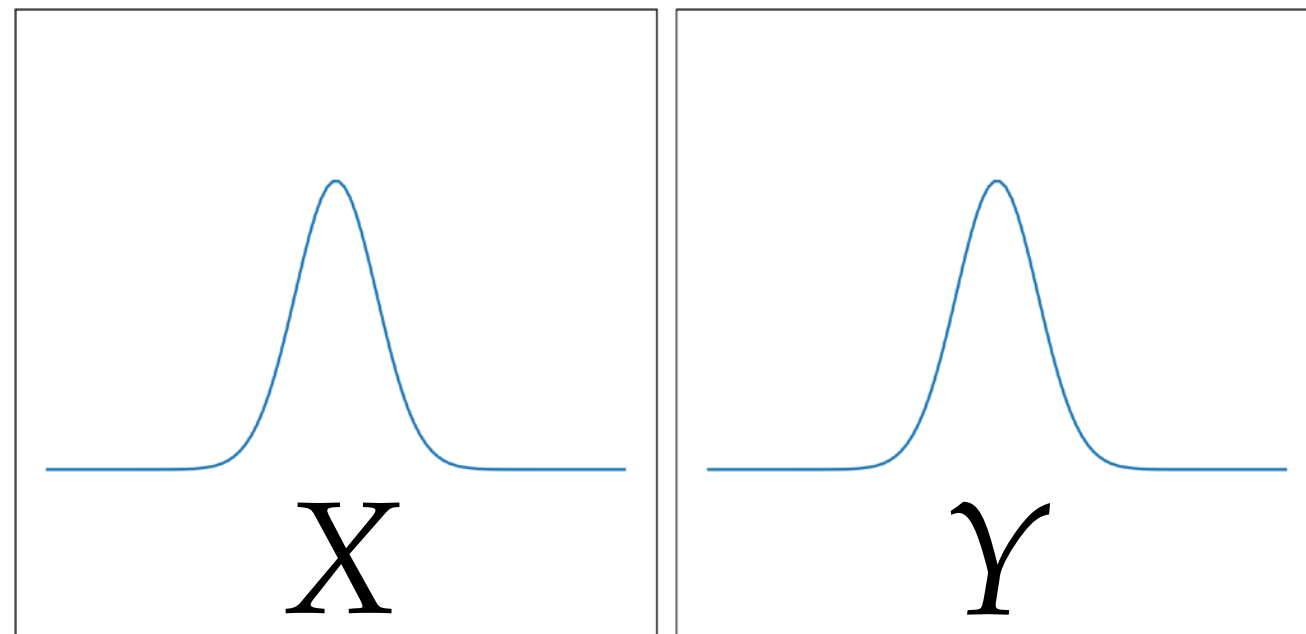
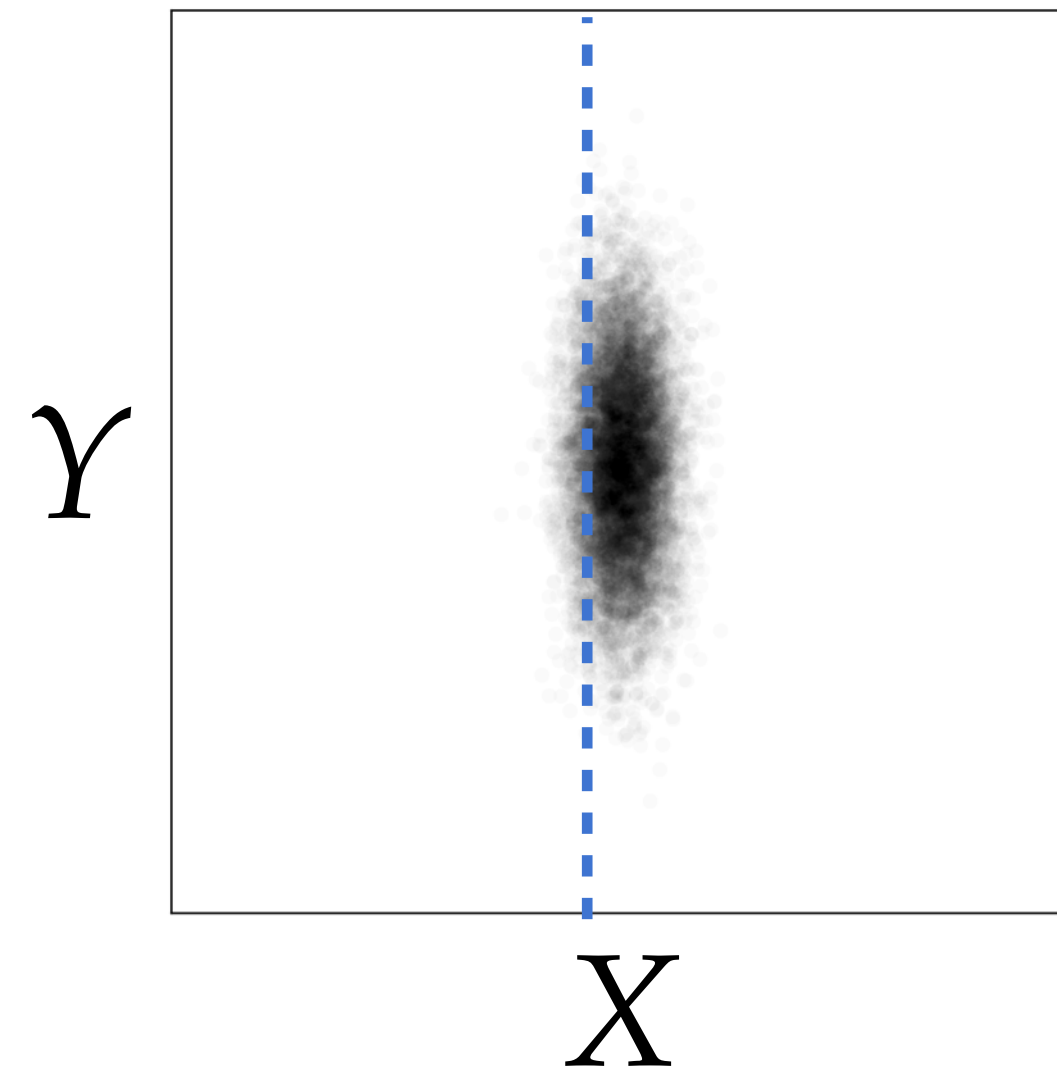
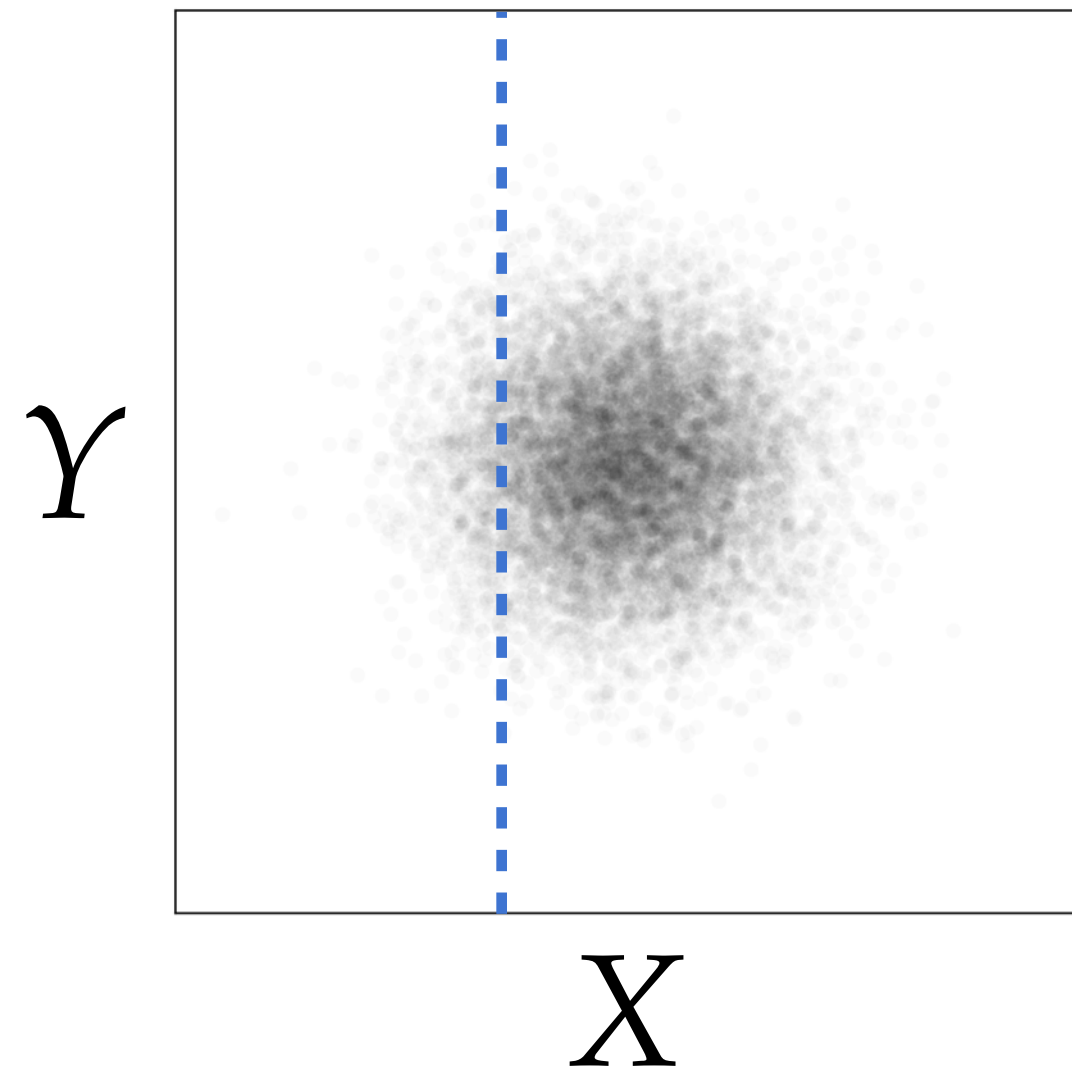
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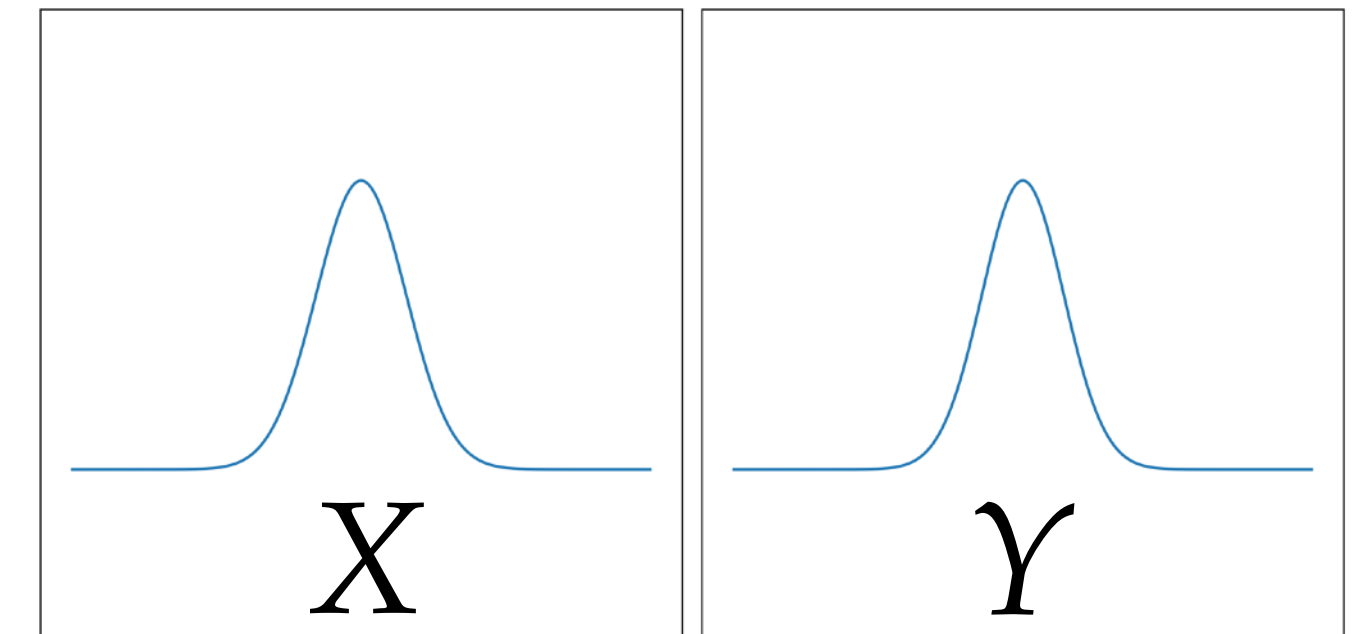
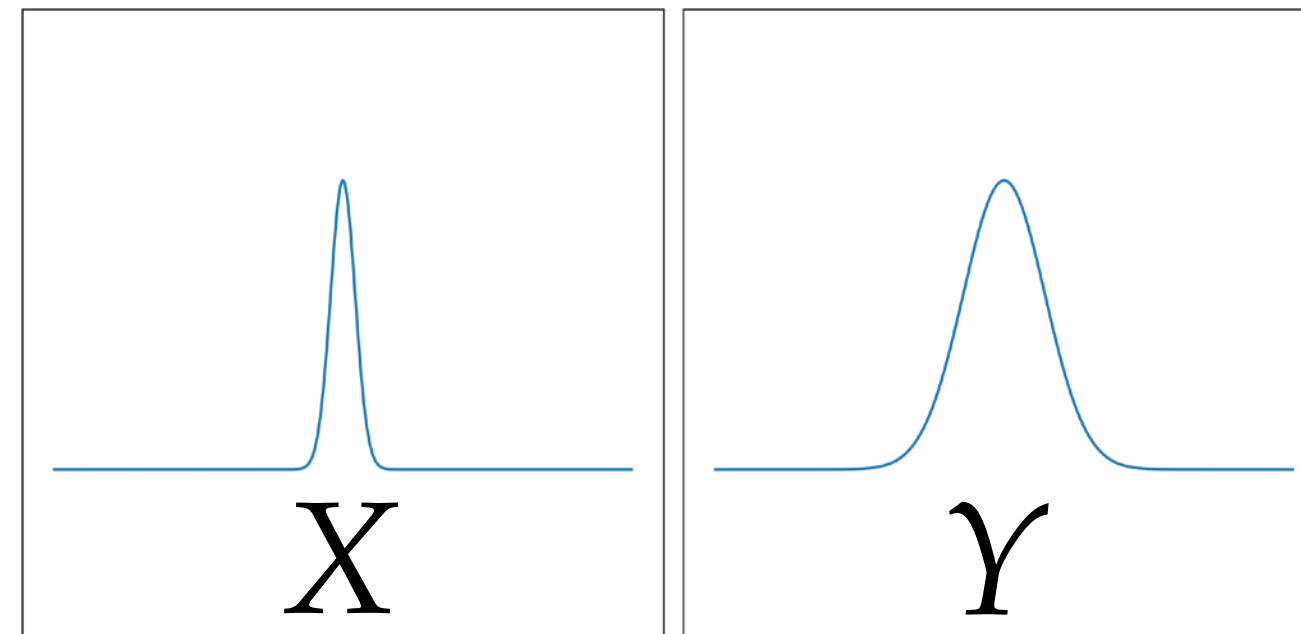
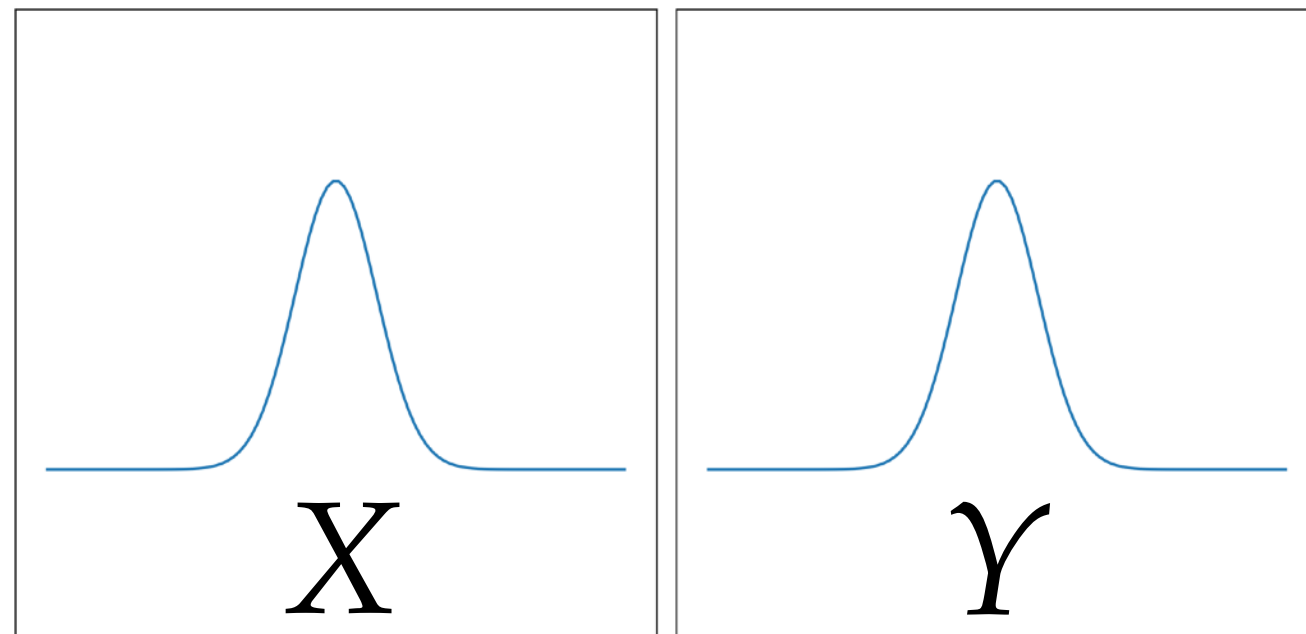
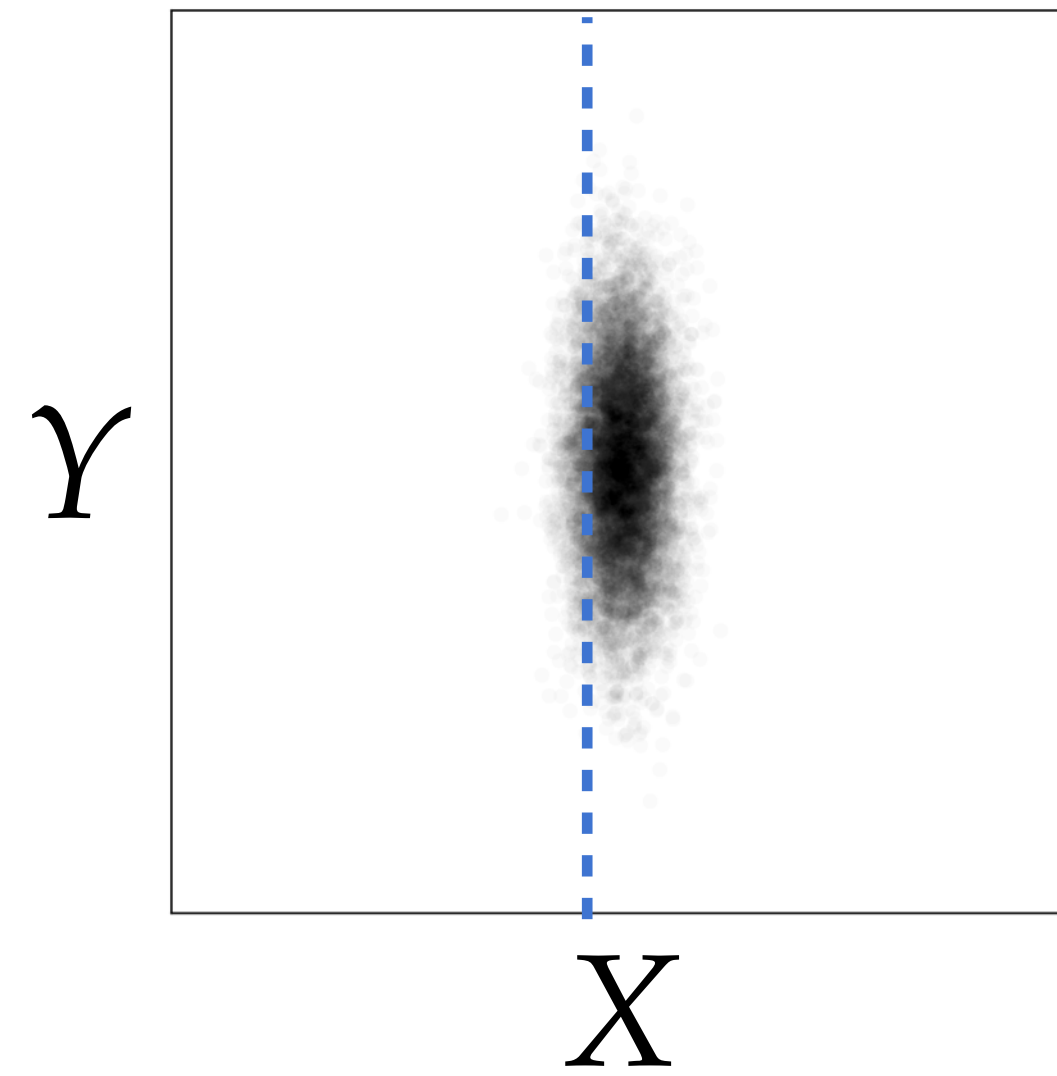
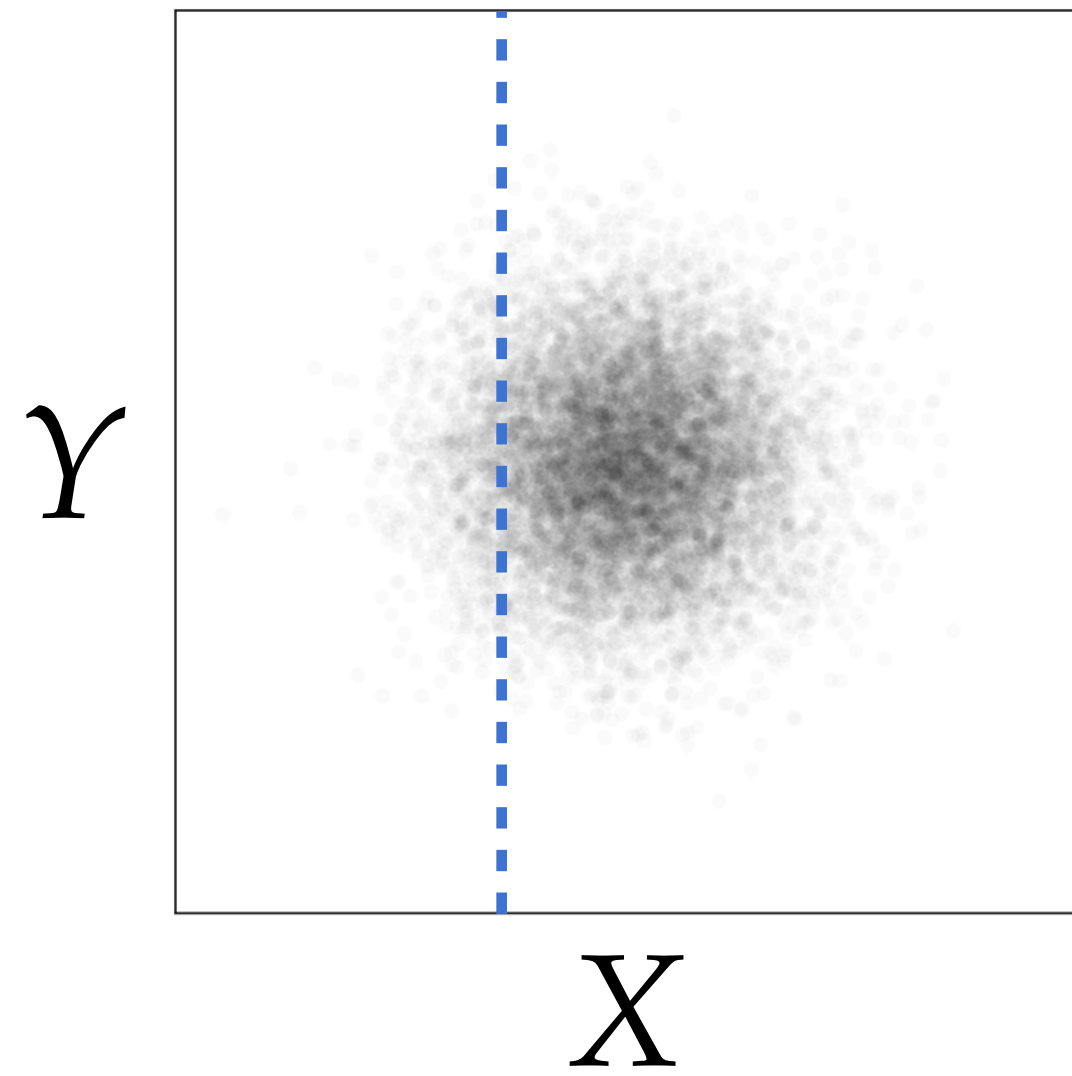
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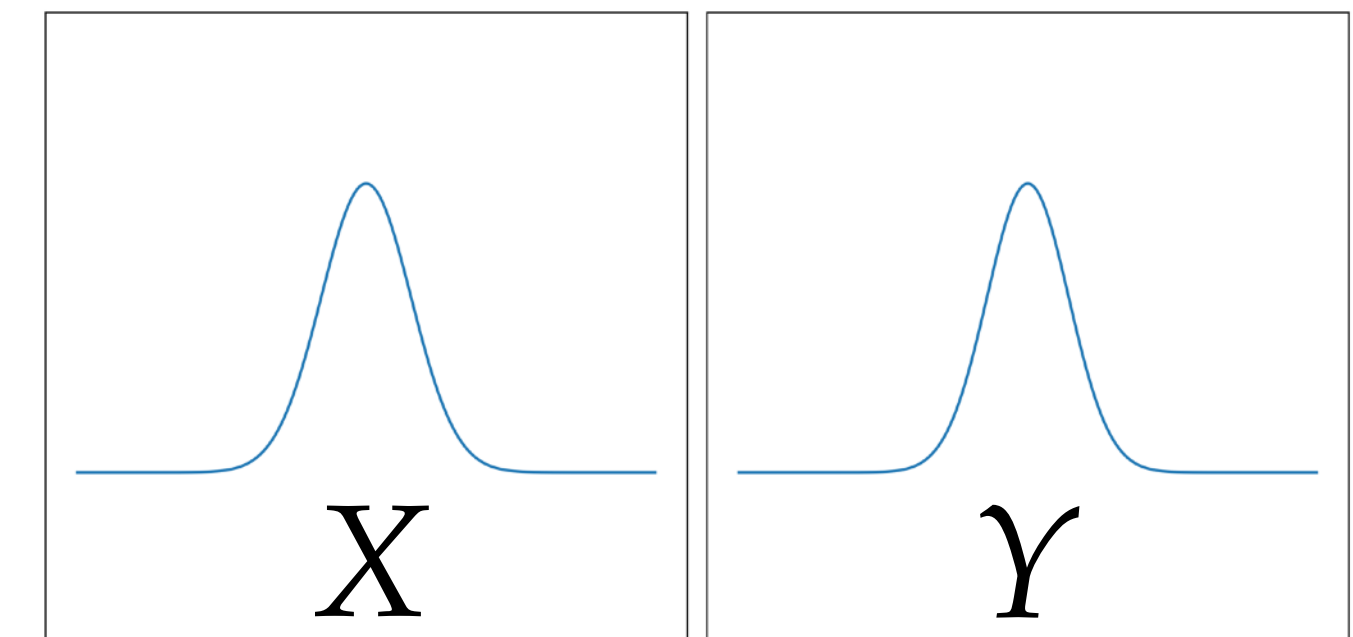
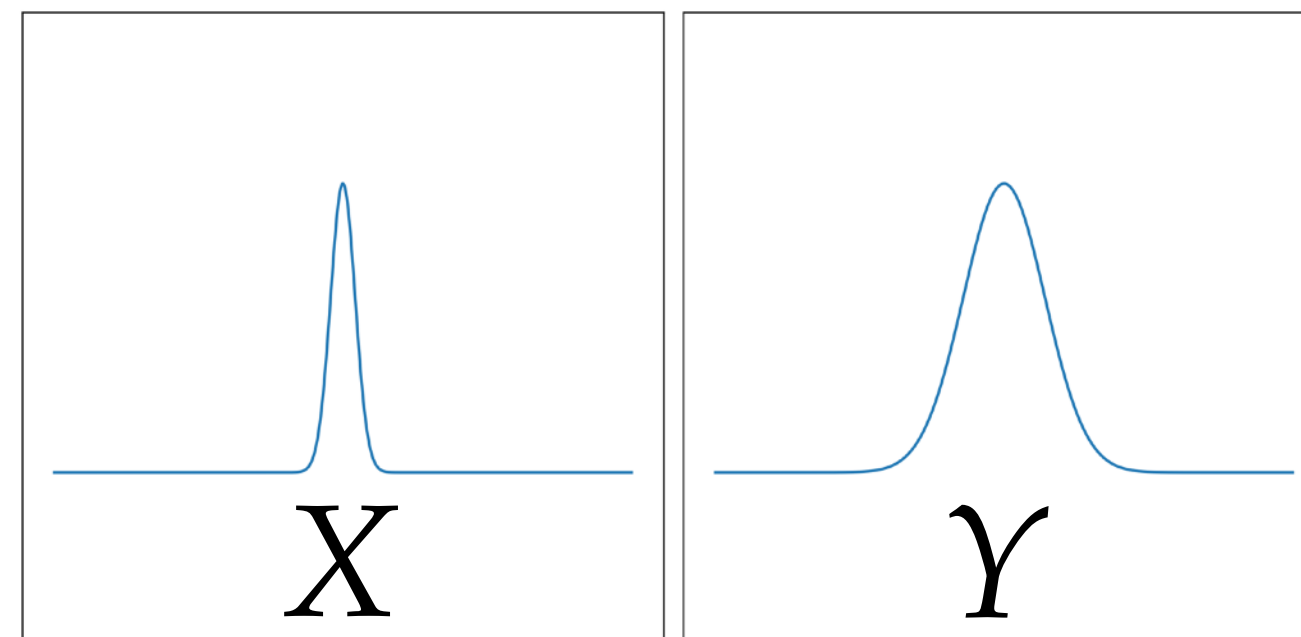
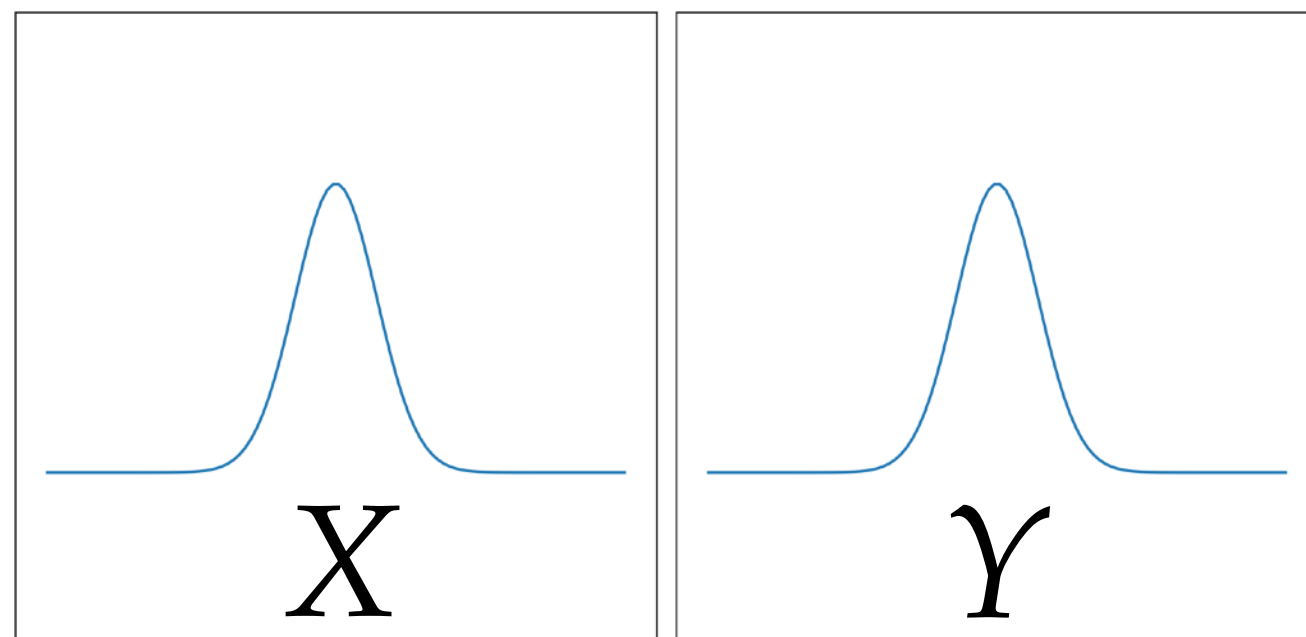
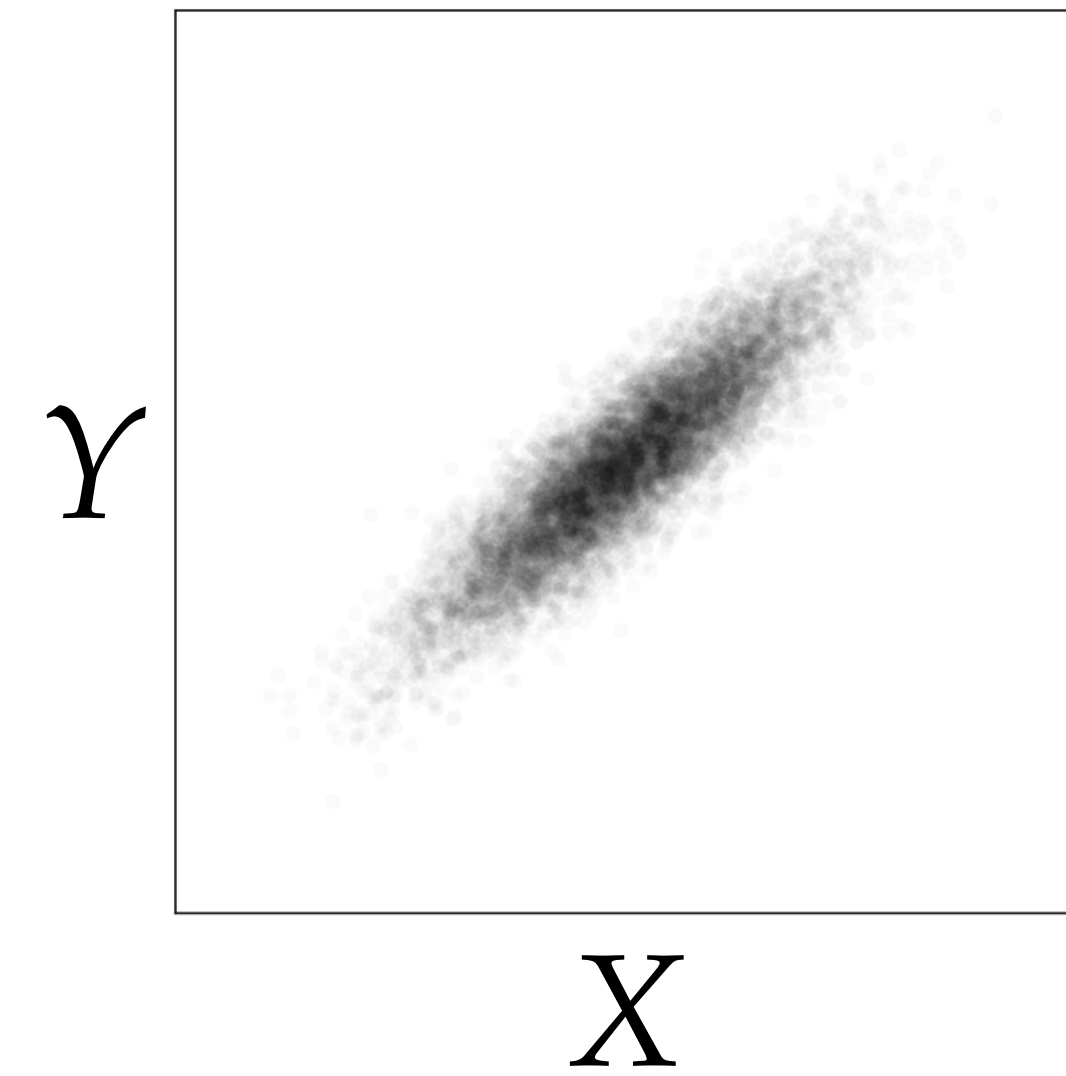
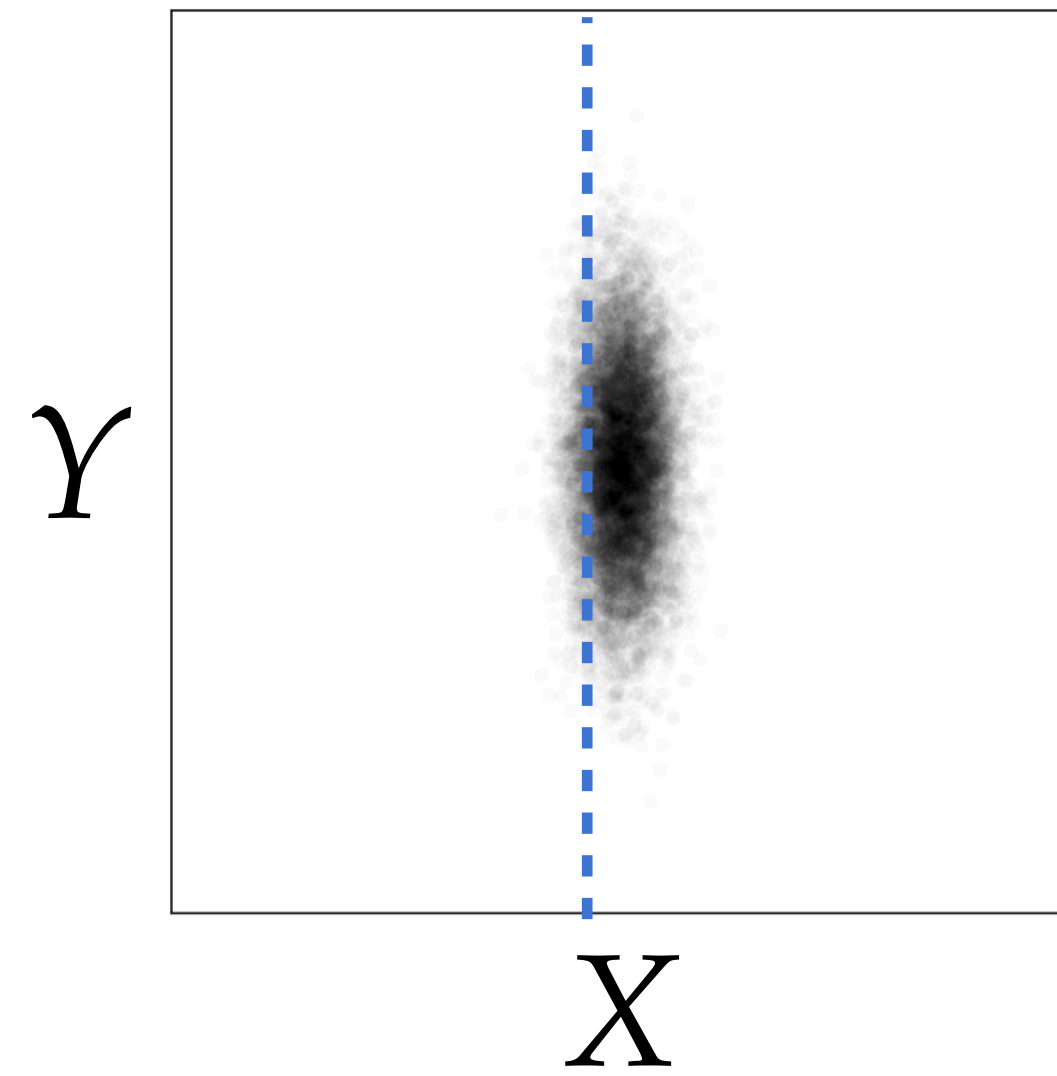
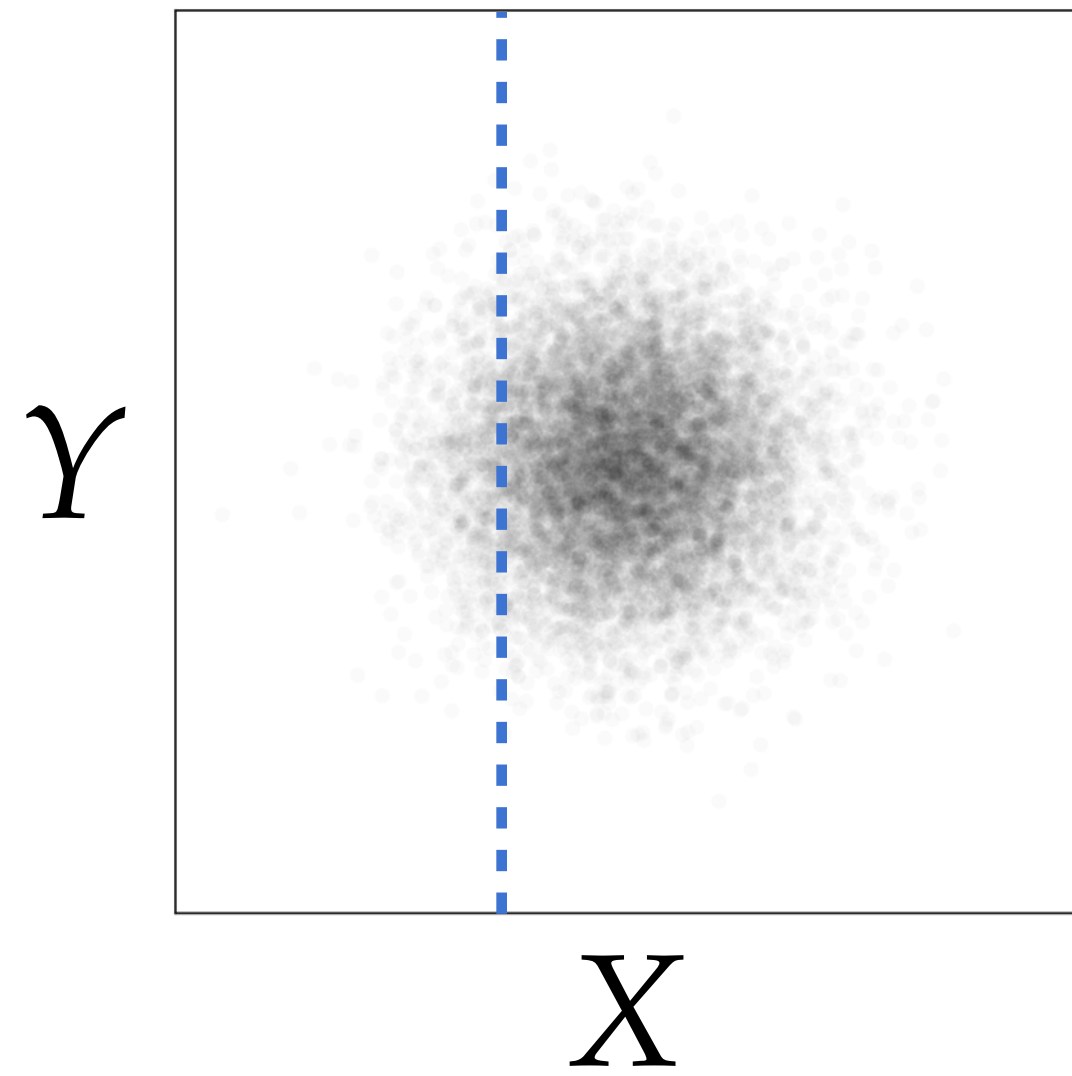
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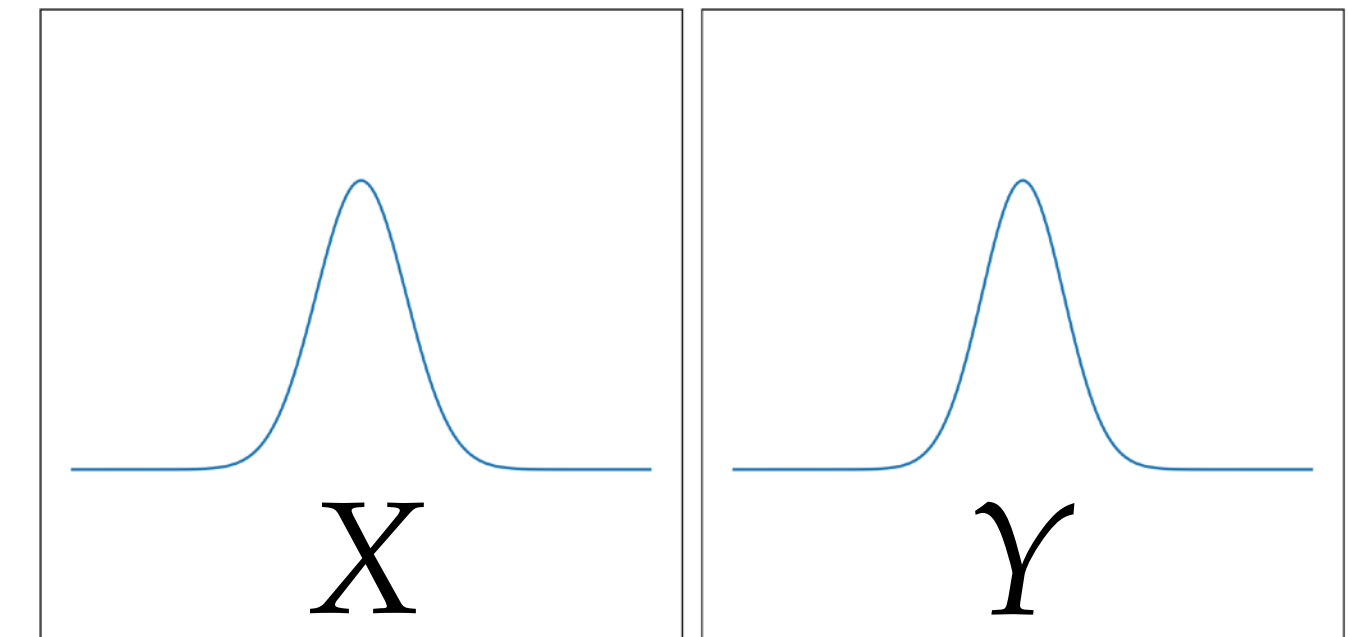
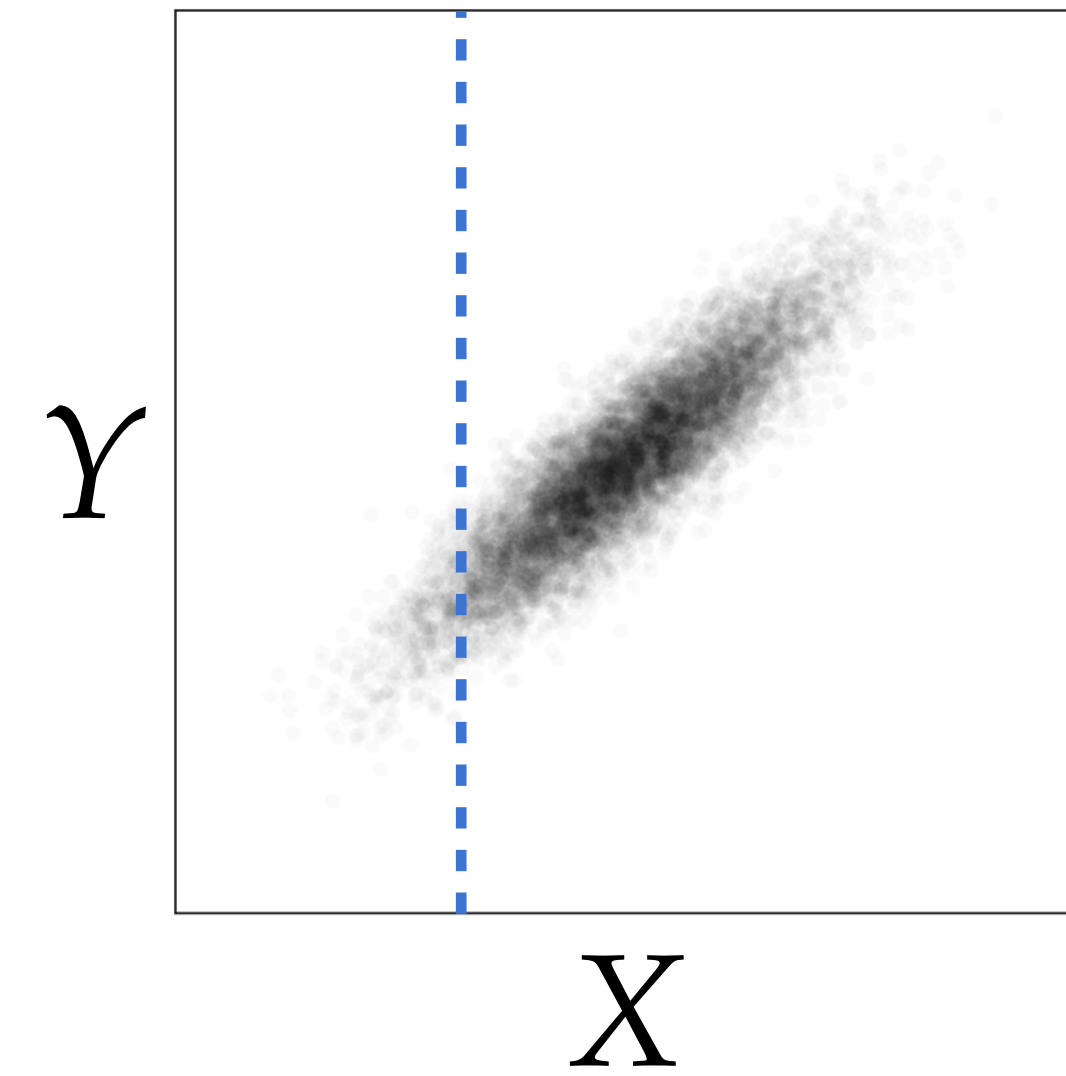
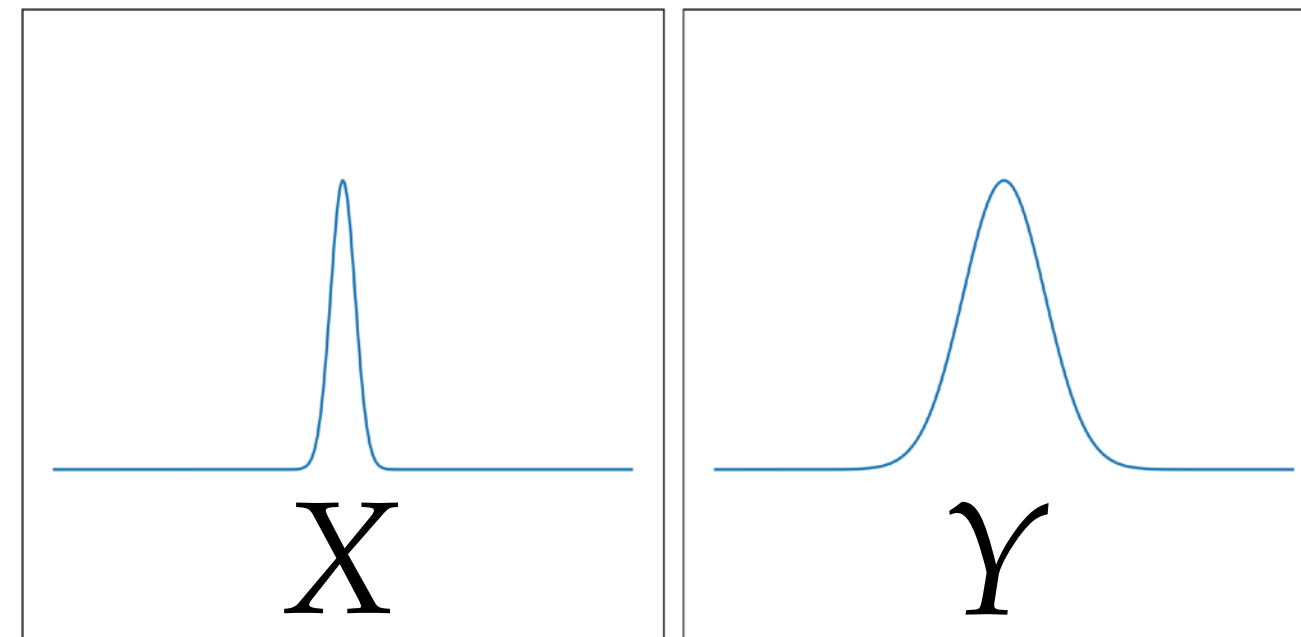
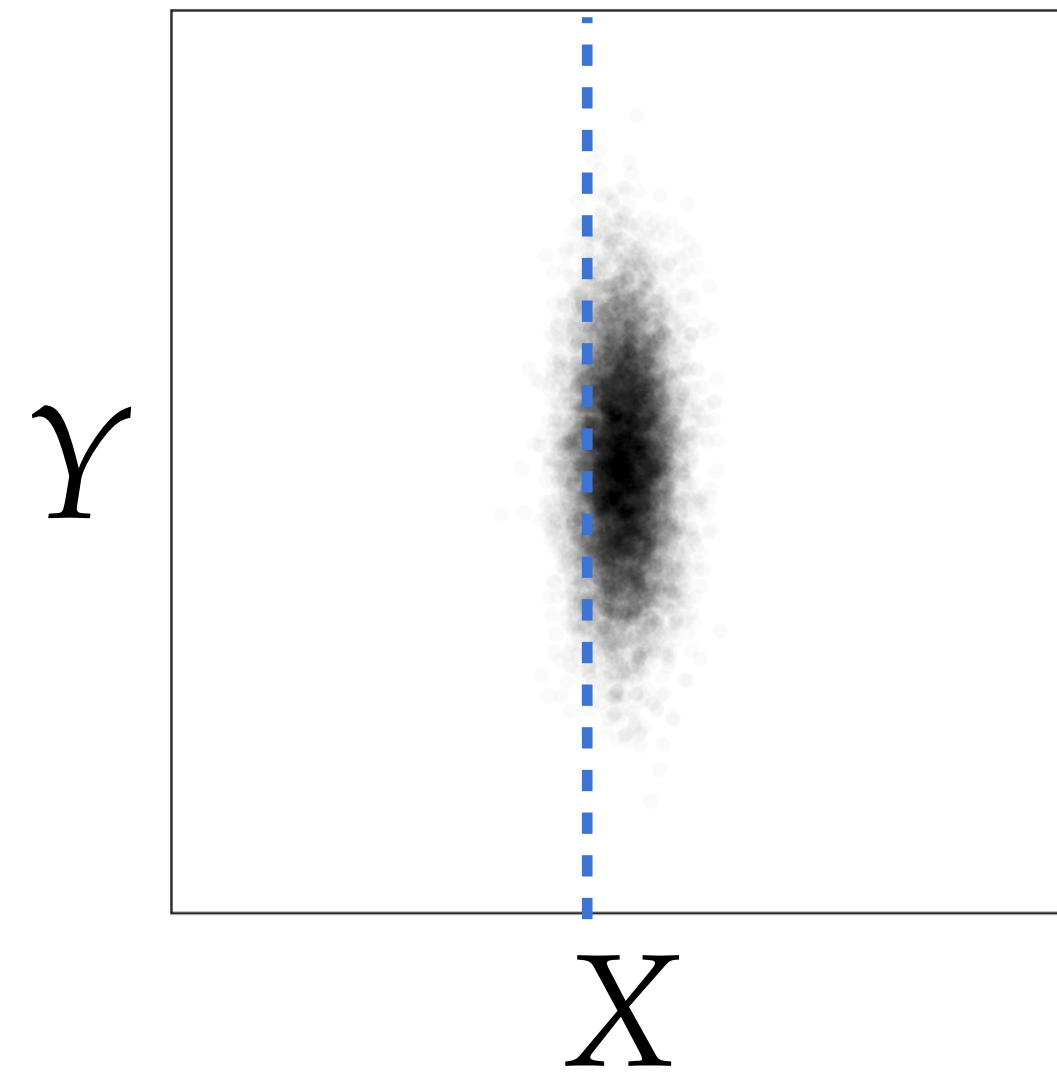
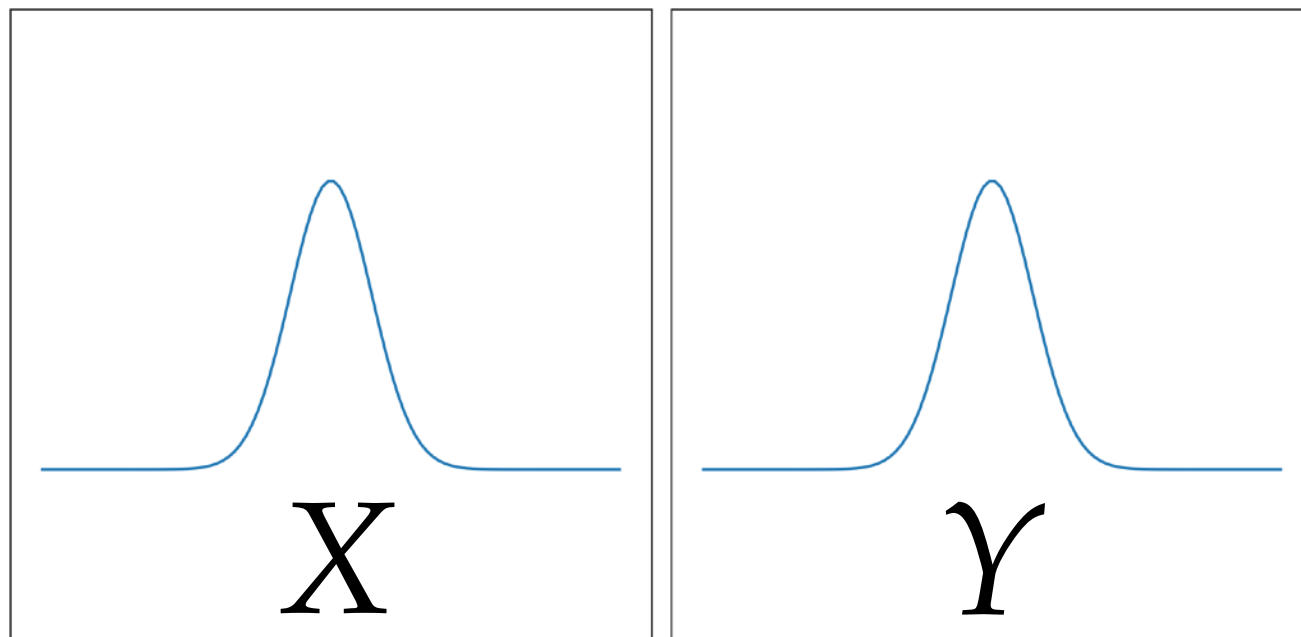
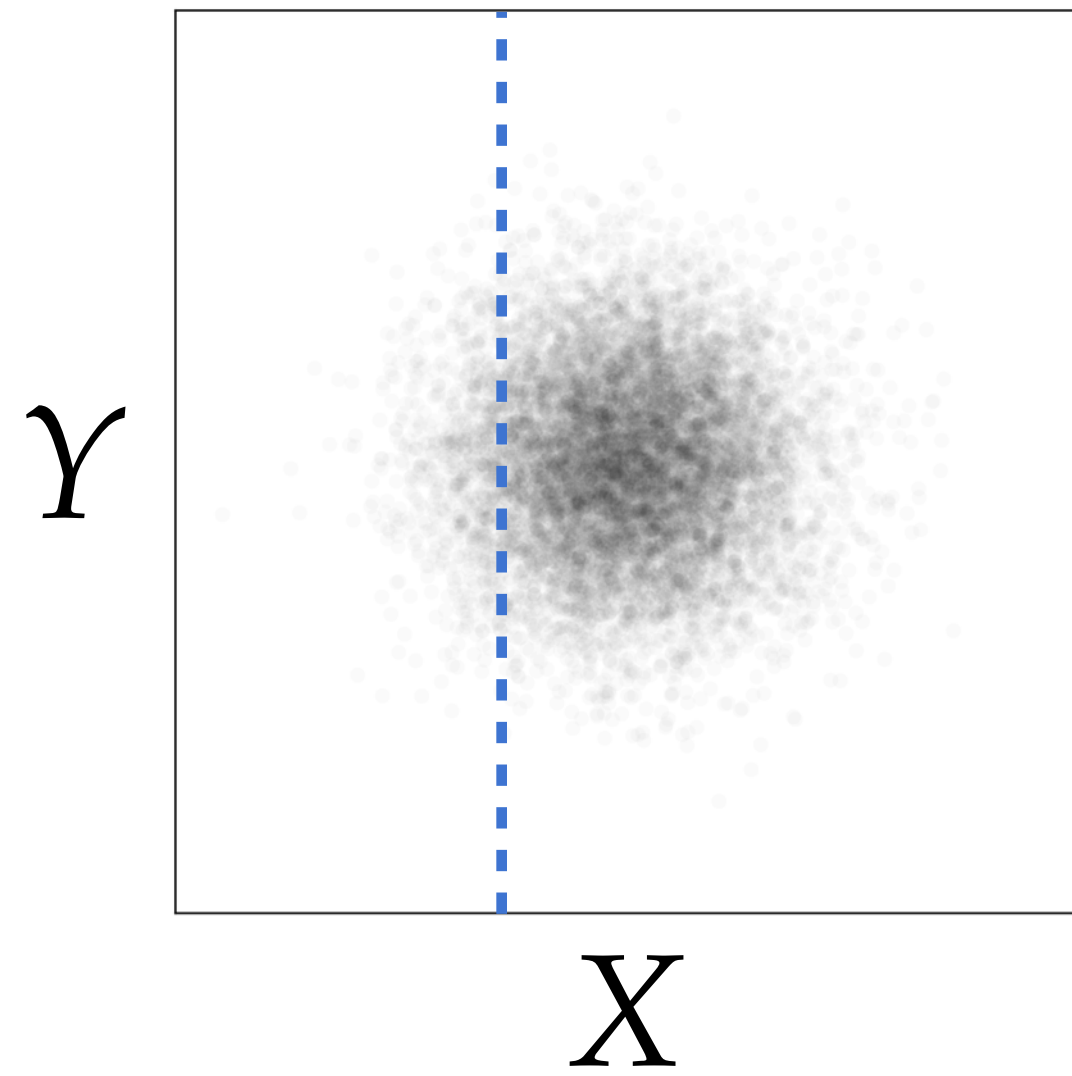
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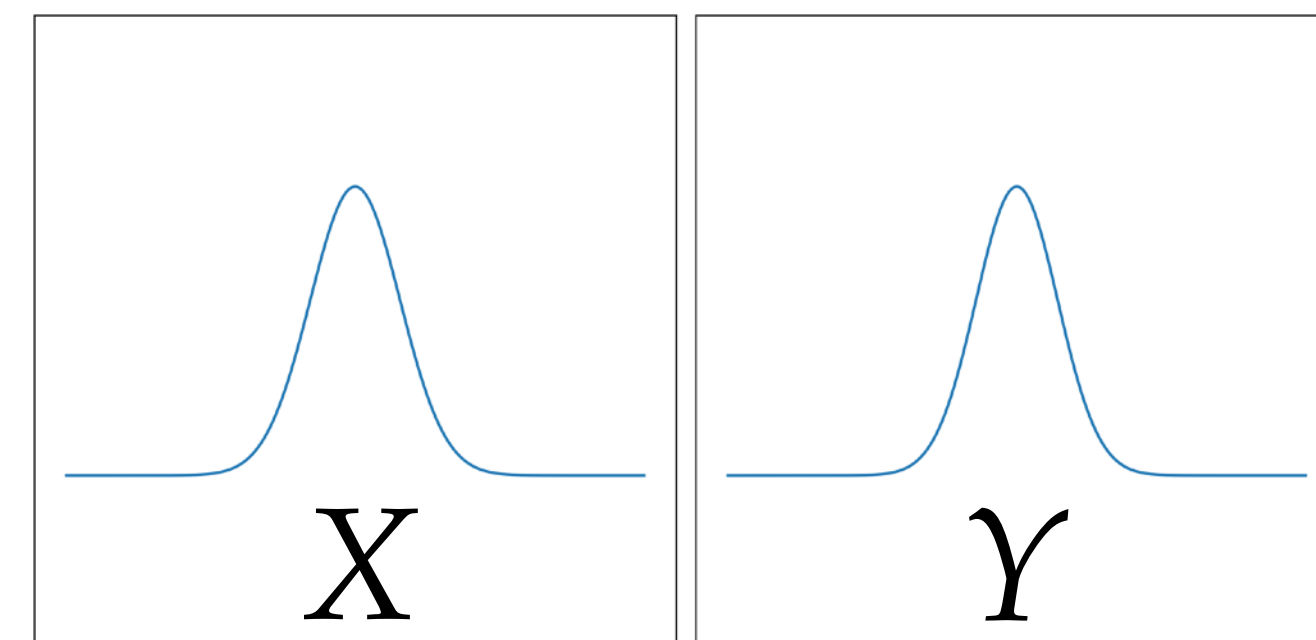
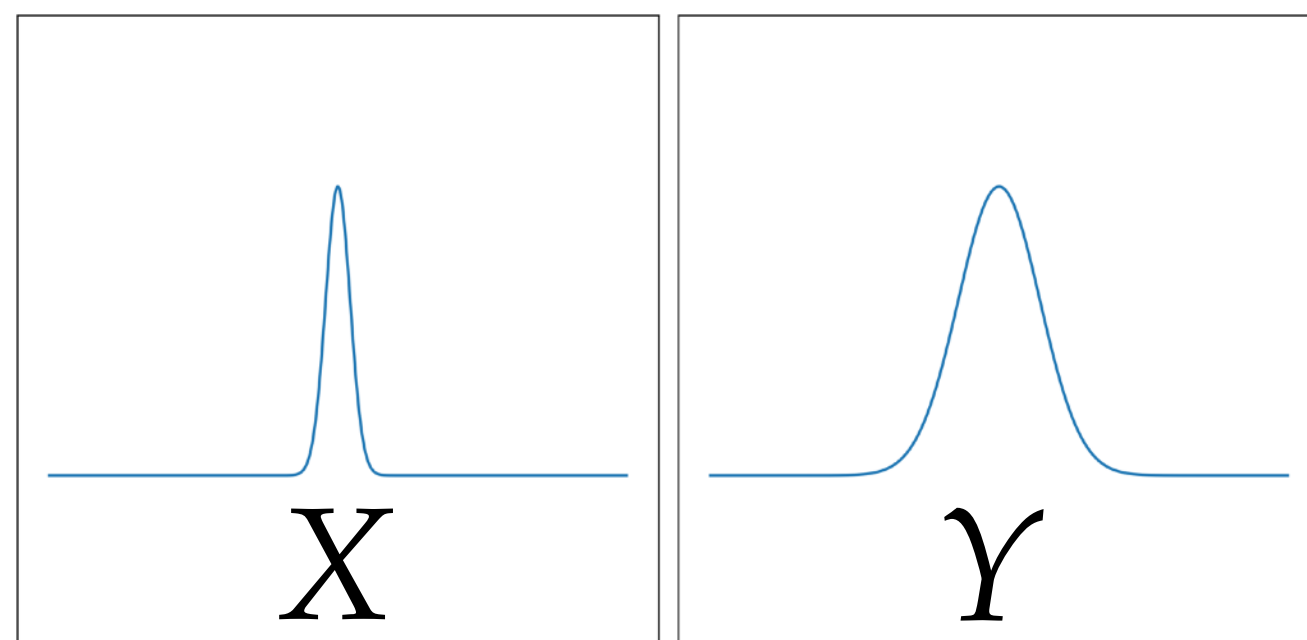
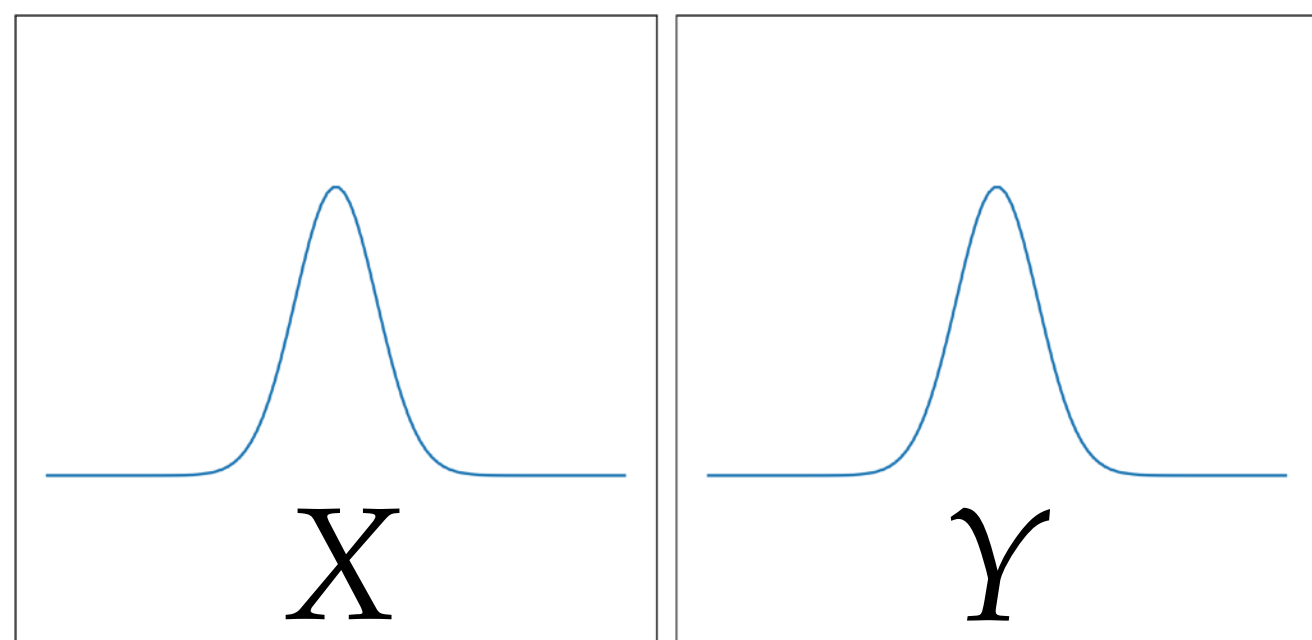
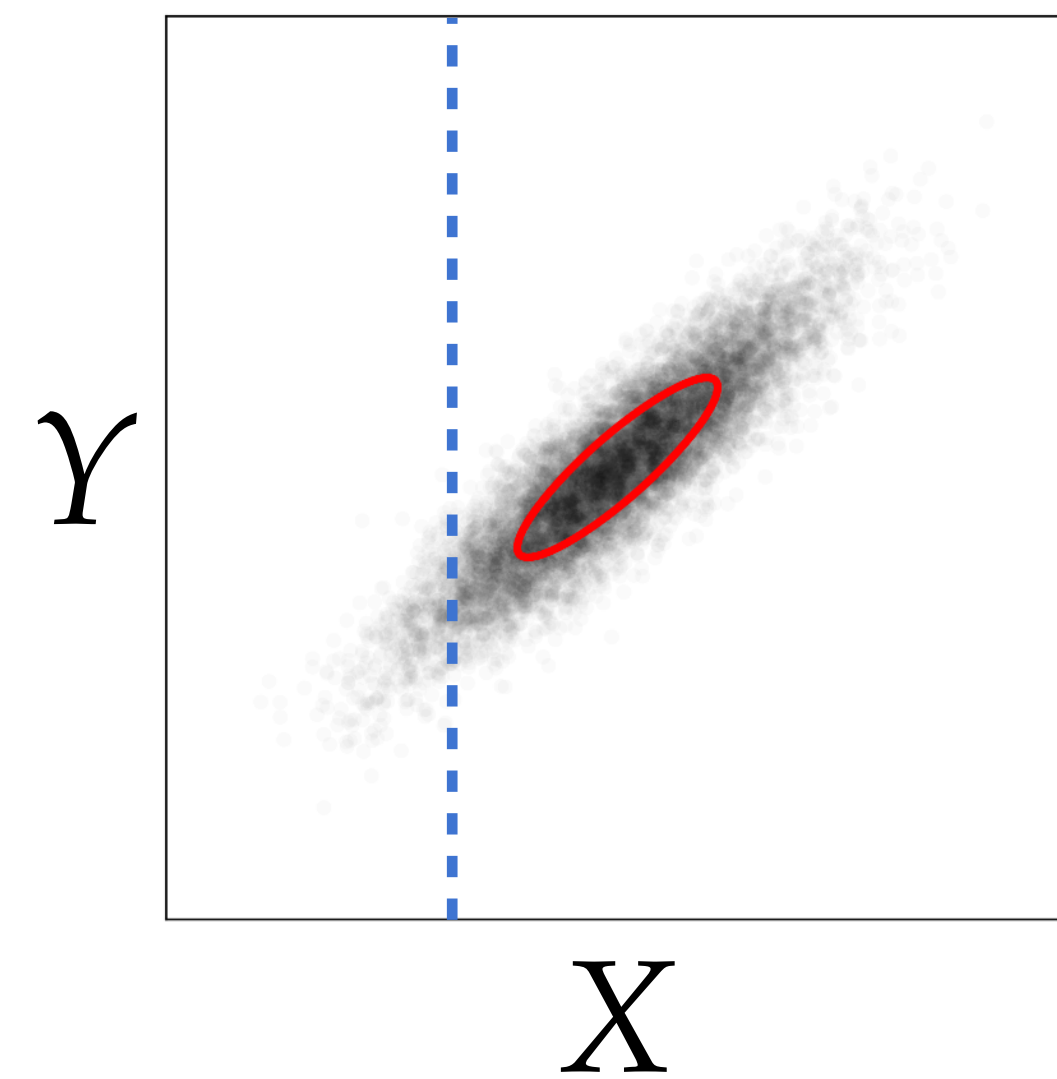
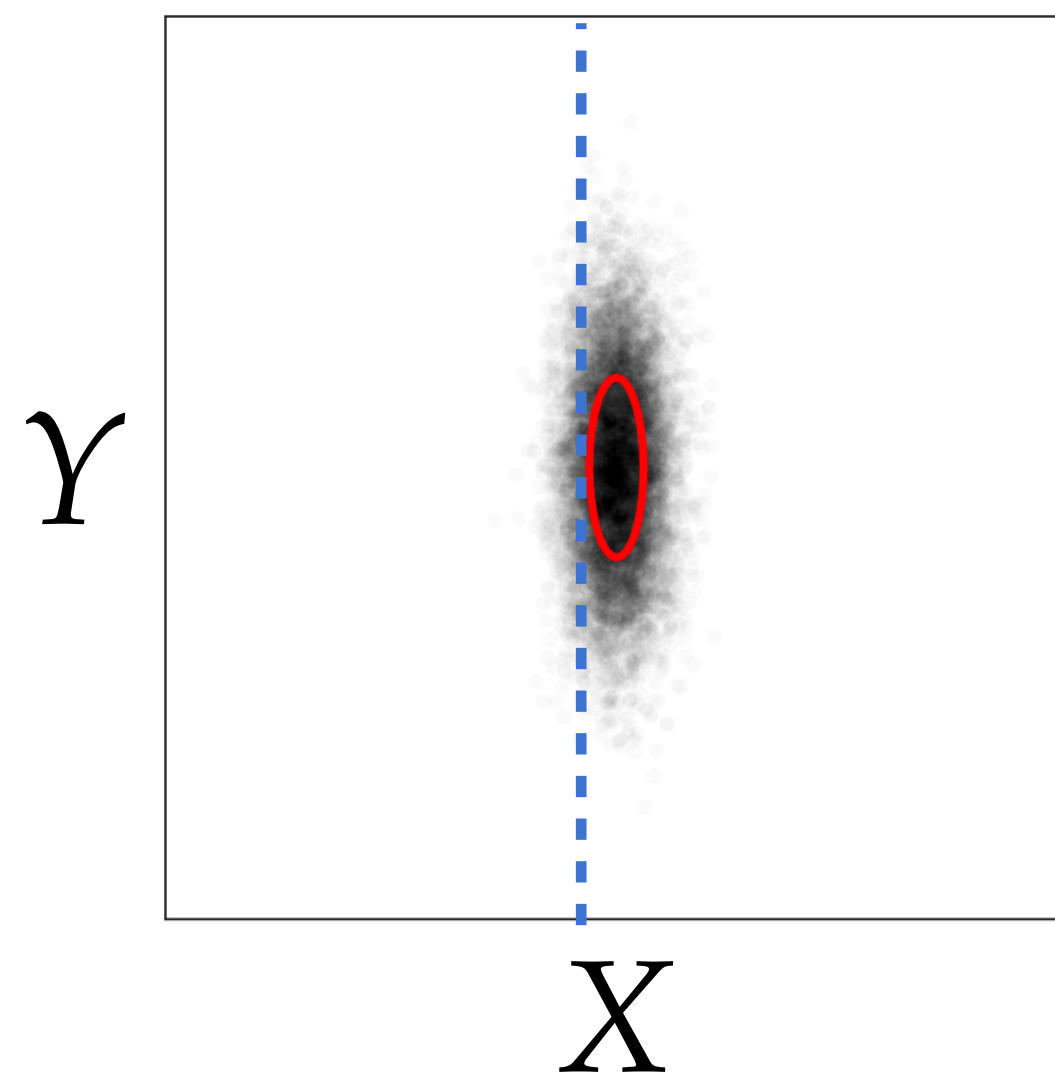
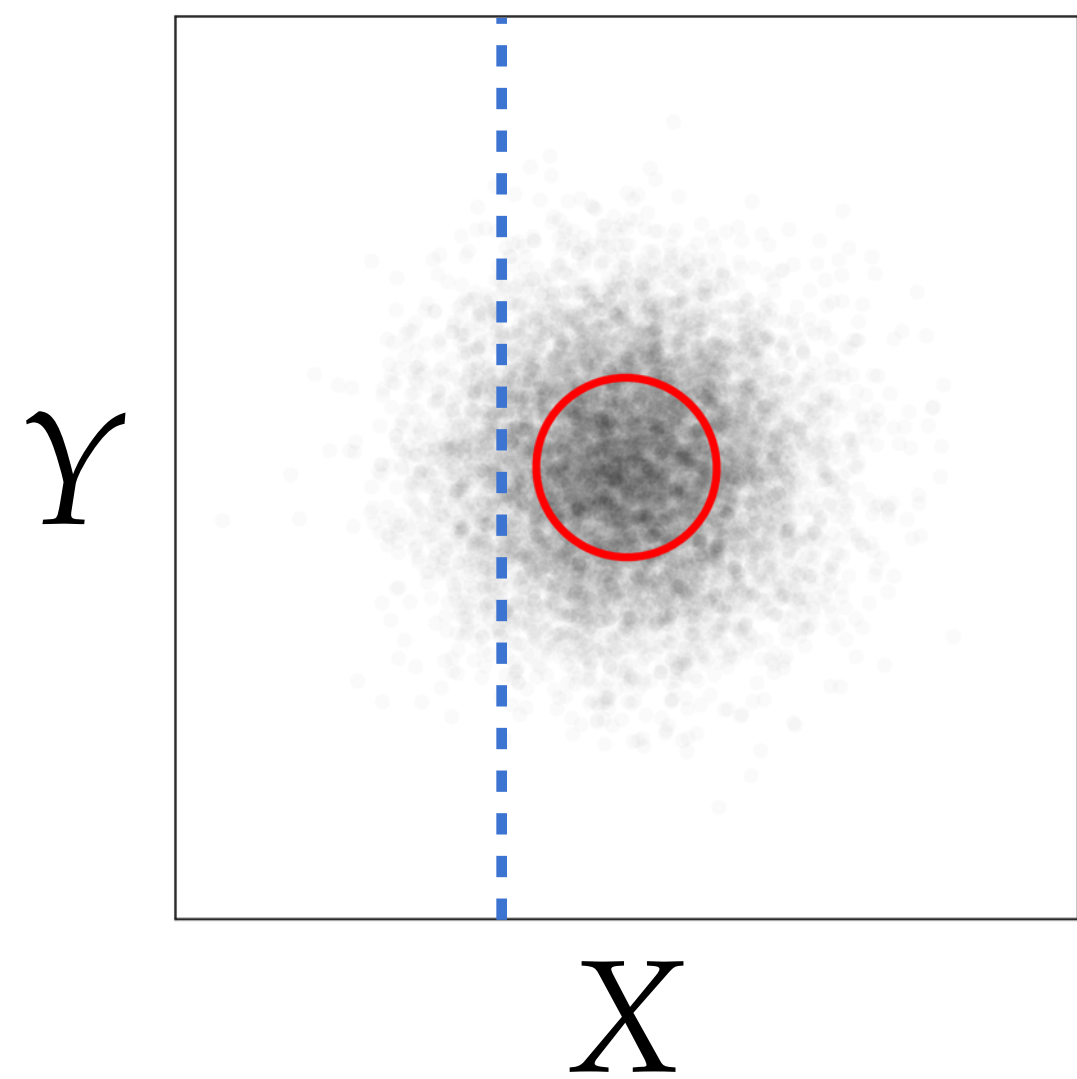
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Correlations in multidimensional data

1. Age
2. Hardness
3. Smell intensity
4. Taste intensity
5. Amount of mold
6. Average color (RGB)



MidJourney: *Board with cheese selection, white background.*

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Intuitively, collecting data will reveal that some fields have an approximately linear relationship (they are **correlated**), e.g.:

- Very old cheeses have a strong taste and are smelly.
- Blue cheeses are generally blue because of mold.

Other axes will show **no correlation** in plots.



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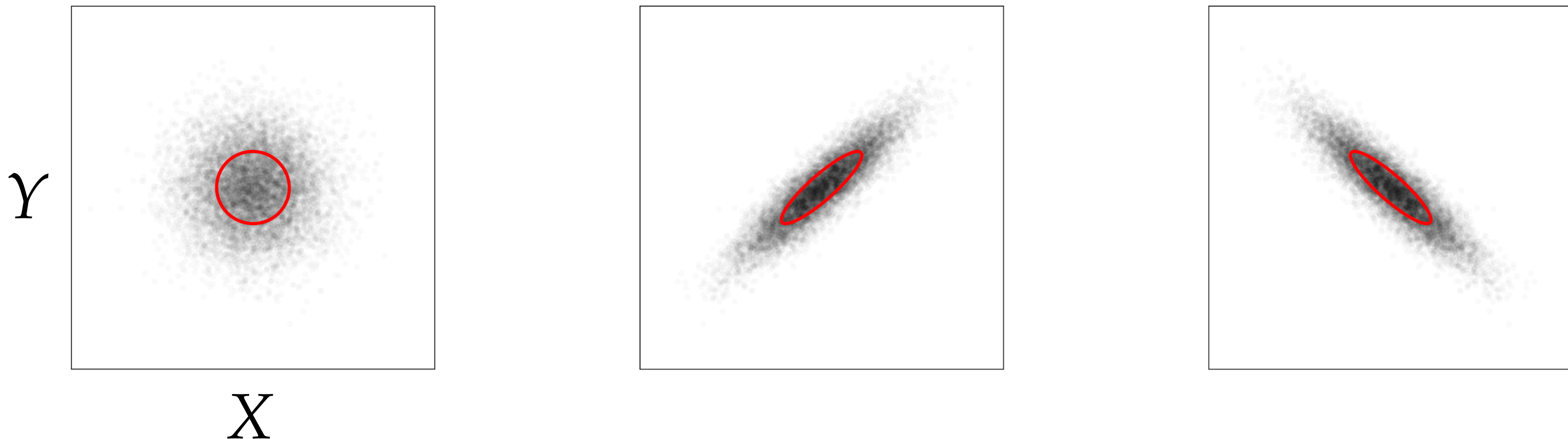
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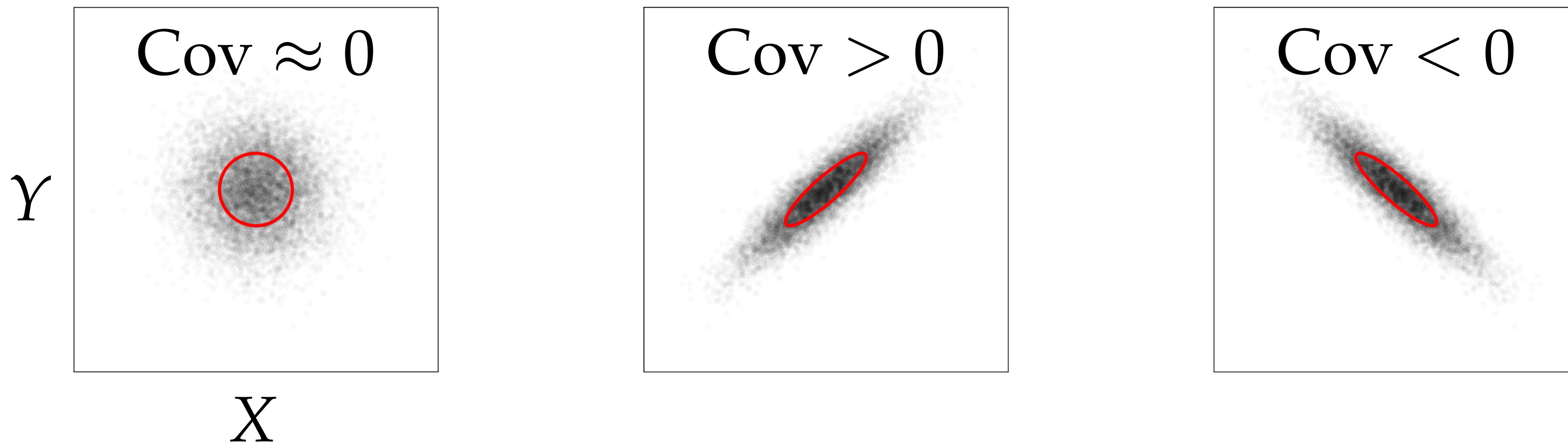
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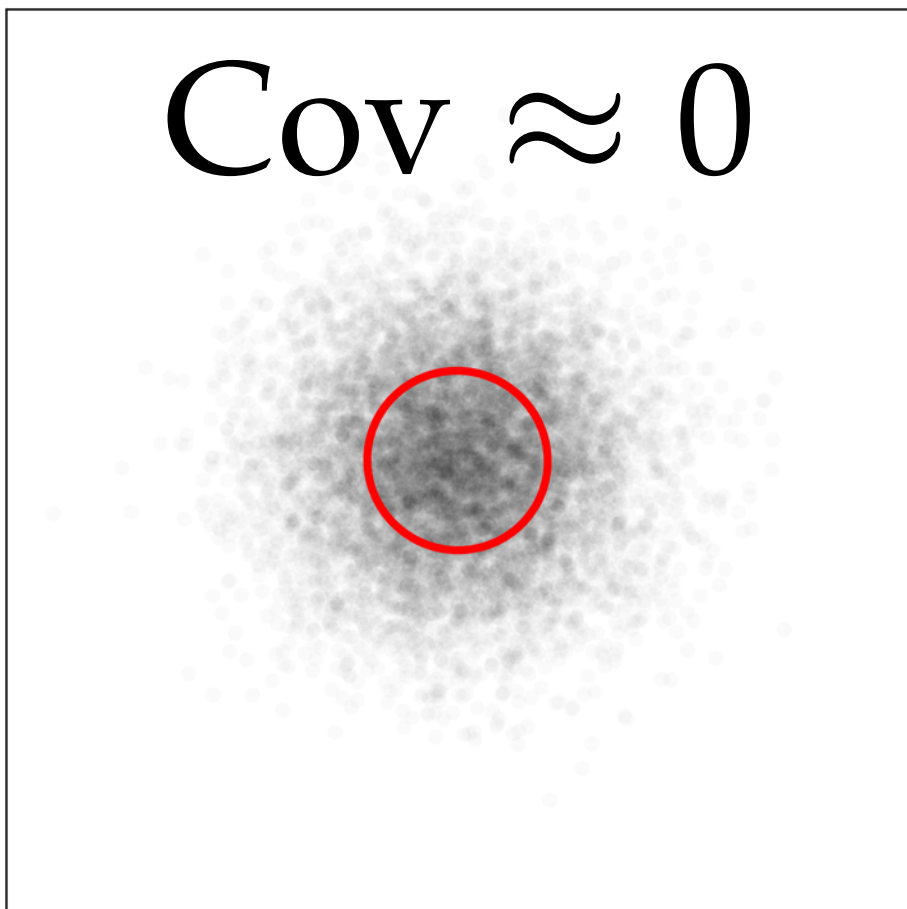


Covariance: in 2 dimensions

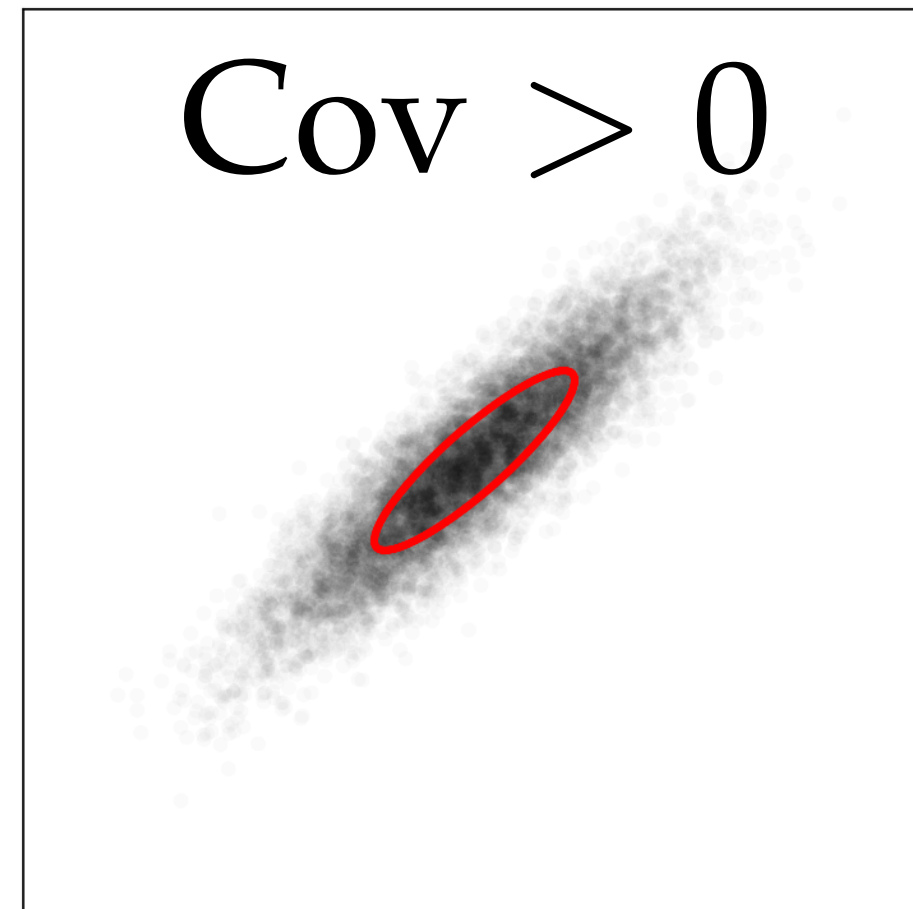
Definition: $\text{Cov}(X, Y) = E [(X - E[X])(Y - E[Y])]$

Estimator: $\overline{(X_k - \bar{X})(Y_k - \bar{Y})} \rightarrow \text{Cov}(X, Y) \quad (k \rightarrow \infty)$

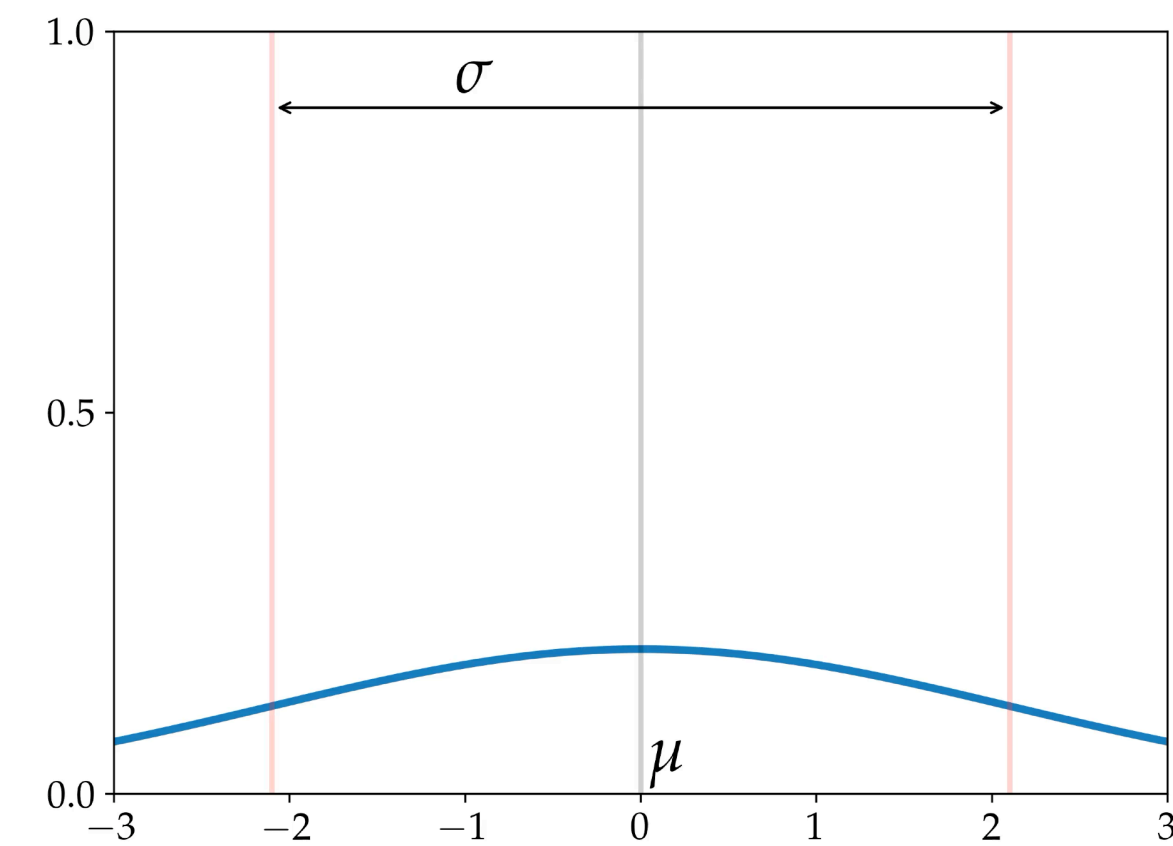
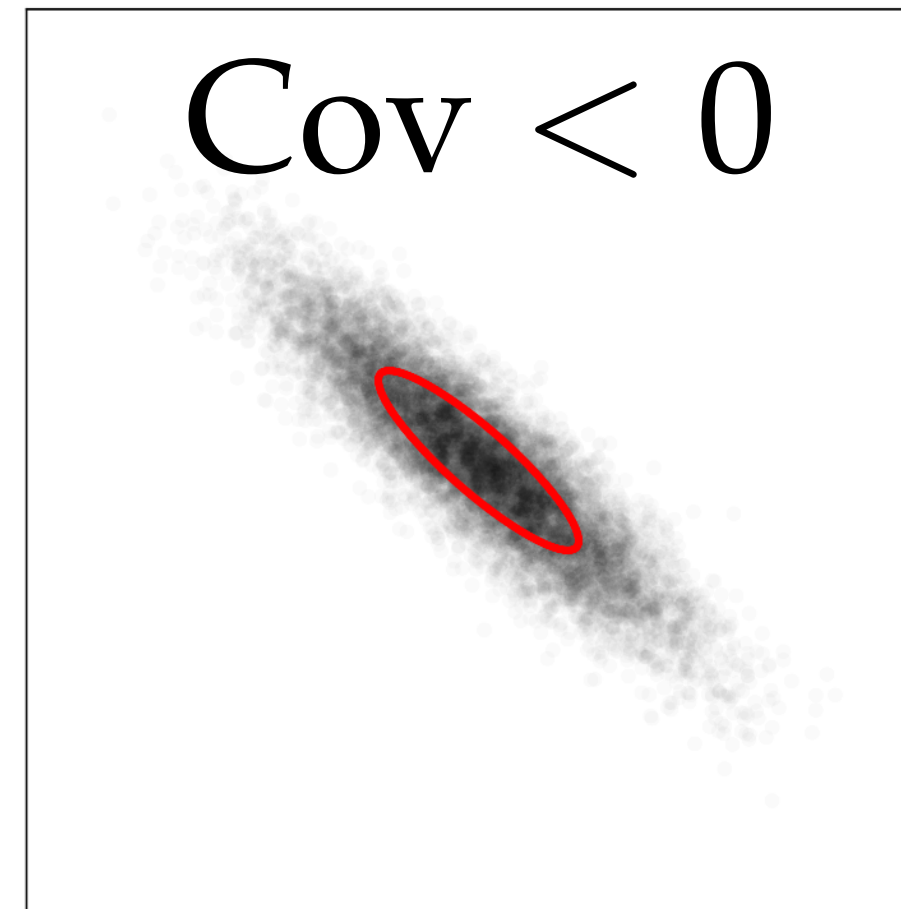
$\text{Cov} \approx 0$



$\text{Cov} > 0$



$\text{Cov} < 0$



X

Covariance: in m dimensions

$$\mathbf{x} = \begin{bmatrix} X_0 - \bar{X} \\ \vdots \\ \vdots \end{bmatrix}$$

Covariance: in m dimensions

$$\mathbf{X} = \begin{bmatrix} X_0 - \bar{X} \\ X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{bmatrix}$$

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$$\Sigma = \frac{1}{N} \mathbf{X}^T \mathbf{X}$$

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Sometimes $N-1$ is used here instead. It depends on the setup (beyond the scope of CS328).

Multivariate Normal Distribution

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

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Covariance matrix

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Mean $\in \mathbb{R}^n$

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Density function:

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normalization constant

exponent

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Creating general normal variates (a.k.a. "simulating" or "sampling" them)

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NumPy recipe: `sqrt_sigma @ np.random.randn(n) + mu`

Statistical Analysis of Multivariate Data

Principal component analysis (PCA): using eigenvectors and eigenvalues to describe data.

- **Step 1:** compute the mean of all data points. Subtract this value from all data points.

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- *Version 2* (via SVD):

- **Step 2:** compute $\mathbf{U}, \mathbf{S}, \mathbf{V} = \text{svd}(\mathbf{X} / \sqrt{n})$
- **Profit:** principal directions given by \mathbf{V} , radii given by \mathbf{S}

Why does this work?

- Covariance matrix is **symmetric**
 - MATH-111: It is therefore a linear transformation that expresses **scaling** along a set of axes. That's it. (It, e.g., cannot rotate or shear its input)
 - Eigendecomposition: explains symmetric matrices as scaling operation in an orthogonal coordinate system

Demo time

The Outer Product

Also often called a rank-1 matrix.

$$\mathbf{uv}^T = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \dots & u_m v_n \end{bmatrix}$$

Another view of the SVD

$$\begin{bmatrix} \mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{U} \end{bmatrix} \begin{bmatrix} \Sigma \end{bmatrix} \begin{bmatrix} \mathbf{V}^T \end{bmatrix}$$

Another view of the SVD

$$\boxed{\mathbf{A}} = \boxed{\mathbf{U}} \boxed{\Sigma} \boxed{\mathbf{V}^T}$$

where

$$\mathbf{V} = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \quad \mathbf{U} = \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ | & & | \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

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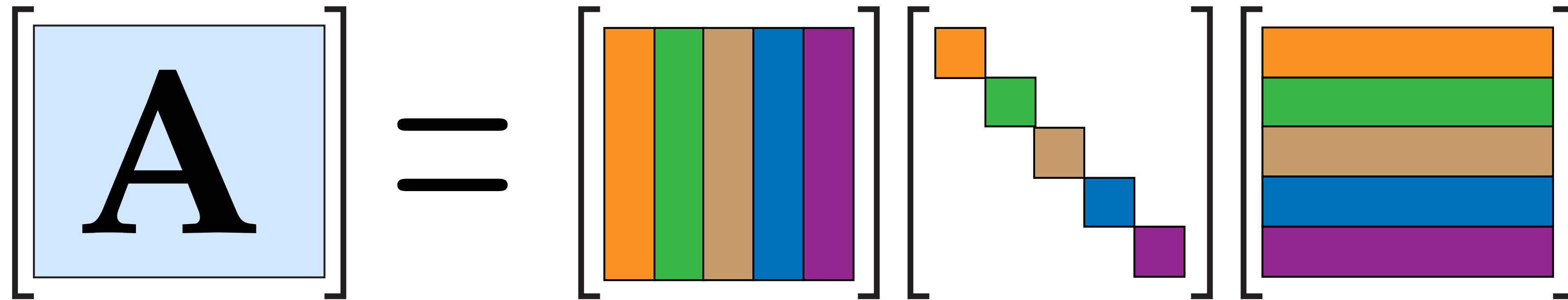
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$$\text{Then } \mathbf{A} = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

Another view of the SVD

Coloring each combination of left/right singular vector & value reveals outer product structure



Another view of the SVD

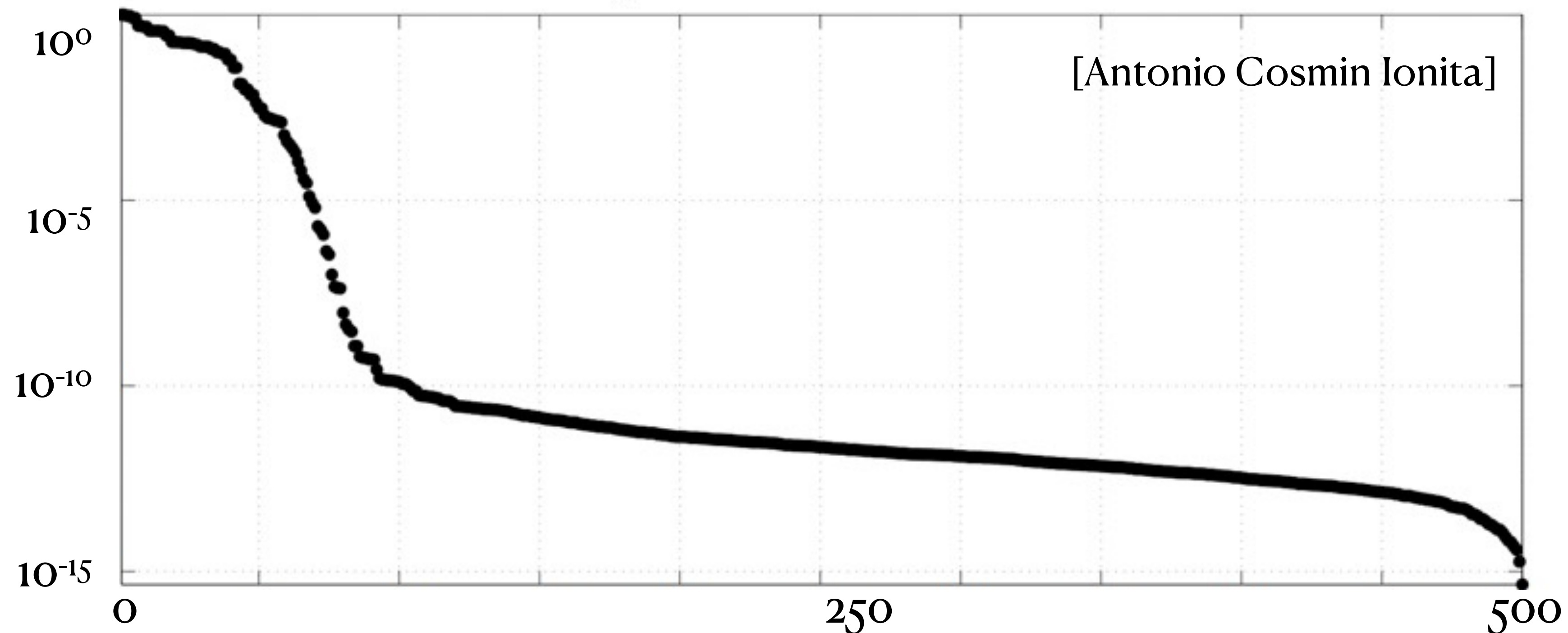
Coloring each combination of left/right singular vector & value reveals outer product structure

$$\begin{bmatrix} \text{A} \end{bmatrix} = \begin{bmatrix} \text{orange} & \text{green} & \text{brown} & \text{blue} & \text{purple} \end{bmatrix} \begin{bmatrix} \text{orange} & & & & \\ & \text{green} & & & \\ & & \text{brown} & & \\ & & & \text{blue} & \\ & & & & \text{purple} \end{bmatrix} \begin{bmatrix} \text{orange} \\ \text{green} \\ \text{brown} \\ \text{blue} \\ \text{purple} \end{bmatrix}$$
$$= \begin{bmatrix} \text{orange} \end{bmatrix} \begin{bmatrix} \text{orange} \end{bmatrix} \begin{bmatrix} \text{orange} \end{bmatrix} + \begin{bmatrix} \text{green} \end{bmatrix} \begin{bmatrix} \text{green} \end{bmatrix} \begin{bmatrix} \text{green} \end{bmatrix} + \begin{bmatrix} \text{brown} \end{bmatrix} \begin{bmatrix} \text{brown} \end{bmatrix} \begin{bmatrix} \text{brown} \end{bmatrix} + \dots$$

Matrix approximation

$$\mathbf{A} = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

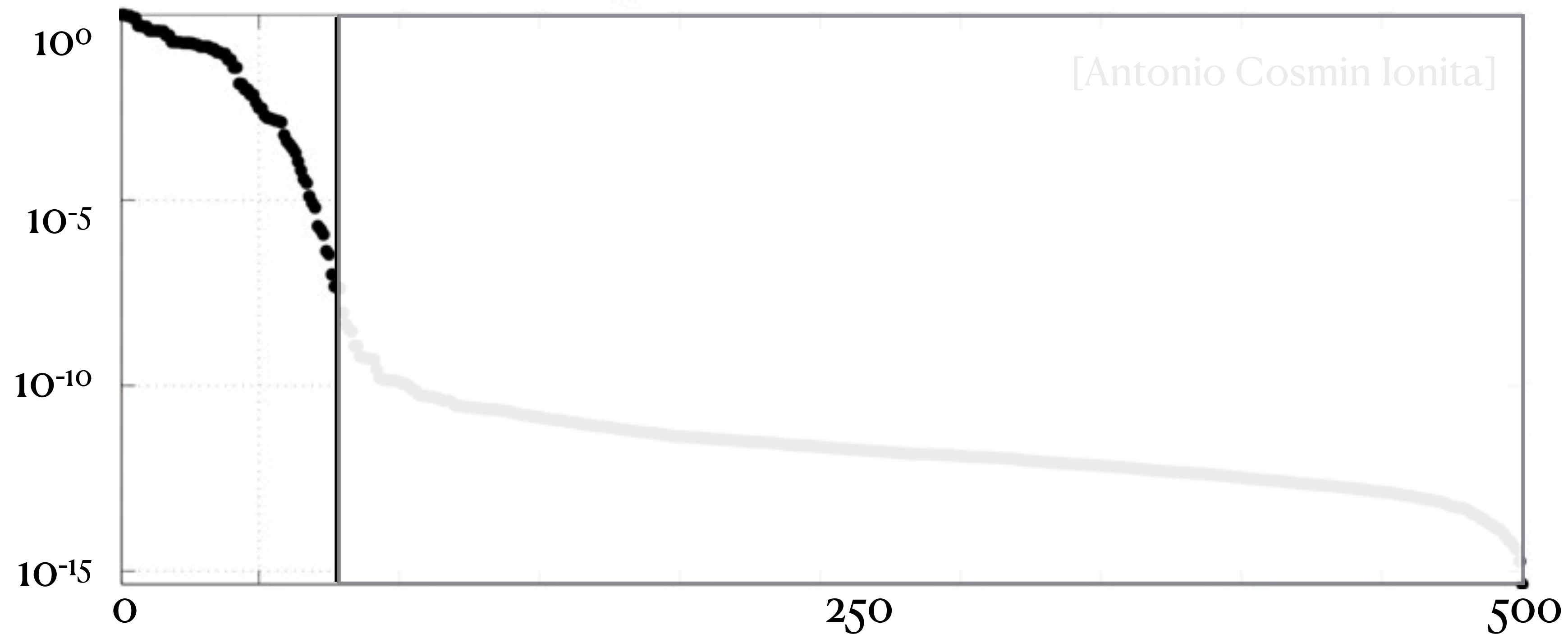
Plot of singular values (in decreasing magnitude)



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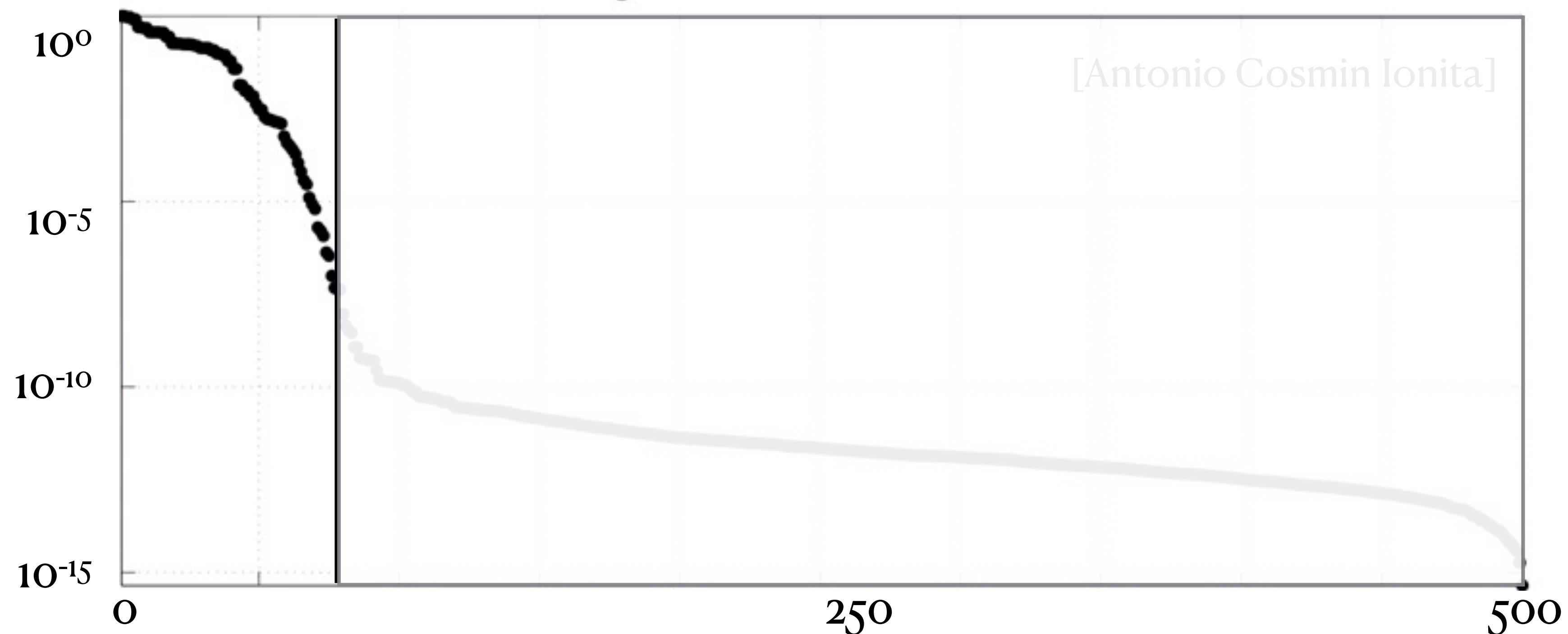
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Matrix approximation

$$\mathbf{A} = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T \longrightarrow \mathbf{A} \approx \sum_{i=1}^{n_{\max}} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

Plot of singular values (in decreasing magnitude)



Eckart-Young Theorem

Stated without proof.

Suppose that \mathbf{A}' is obtained from \mathbf{A} by truncating all but the largest k singular values from its singular value decomposition.

Then \mathbf{A}' minimizes both

$$(i) \quad \|\mathbf{A} - \mathbf{A}'\|_F \quad \text{and}$$

$$(ii) \quad \|\mathbf{A} - \mathbf{A}'\|_2$$

among all matrices \mathbf{A}' with rank k .

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$$\|\mathbf{A}\|_F^2 := \sum_{ij} A_{ij}^2$$

(“Frobenius norm”)

Demo time

Computing inverses via the SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

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Computing inverses via the SVD

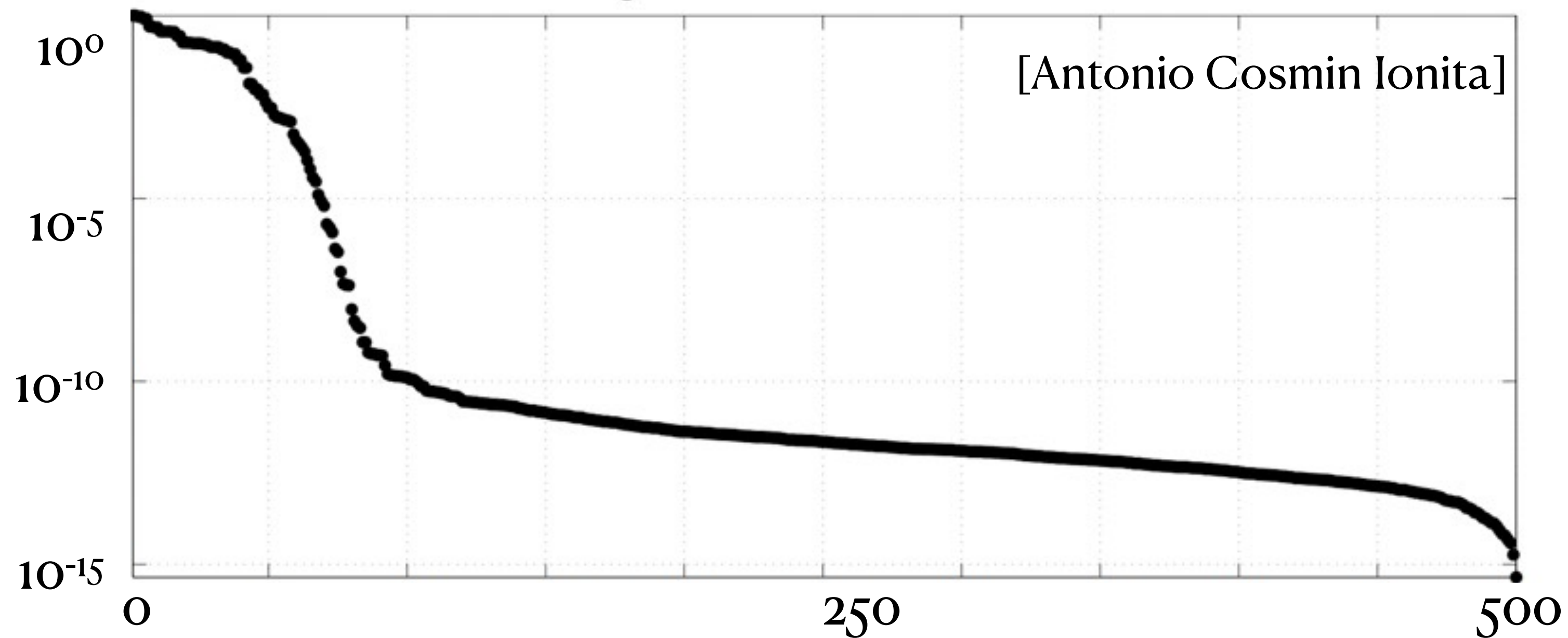
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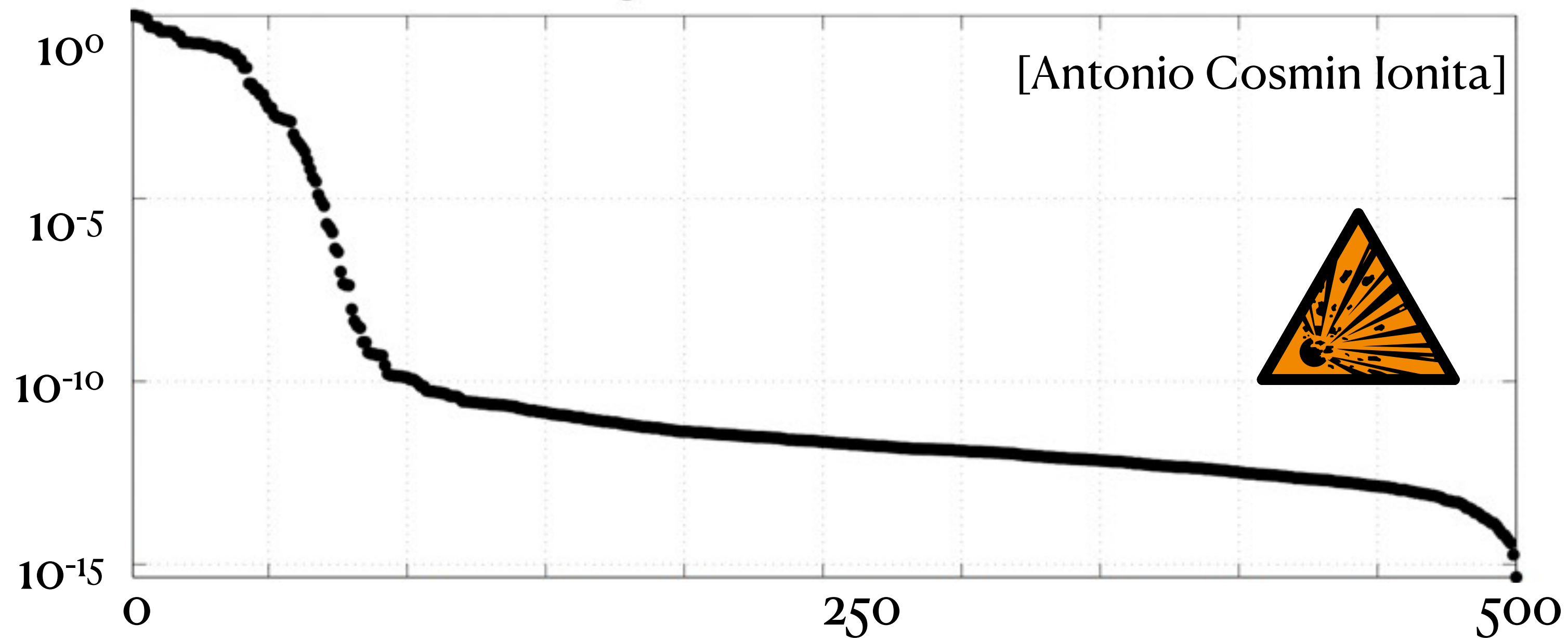
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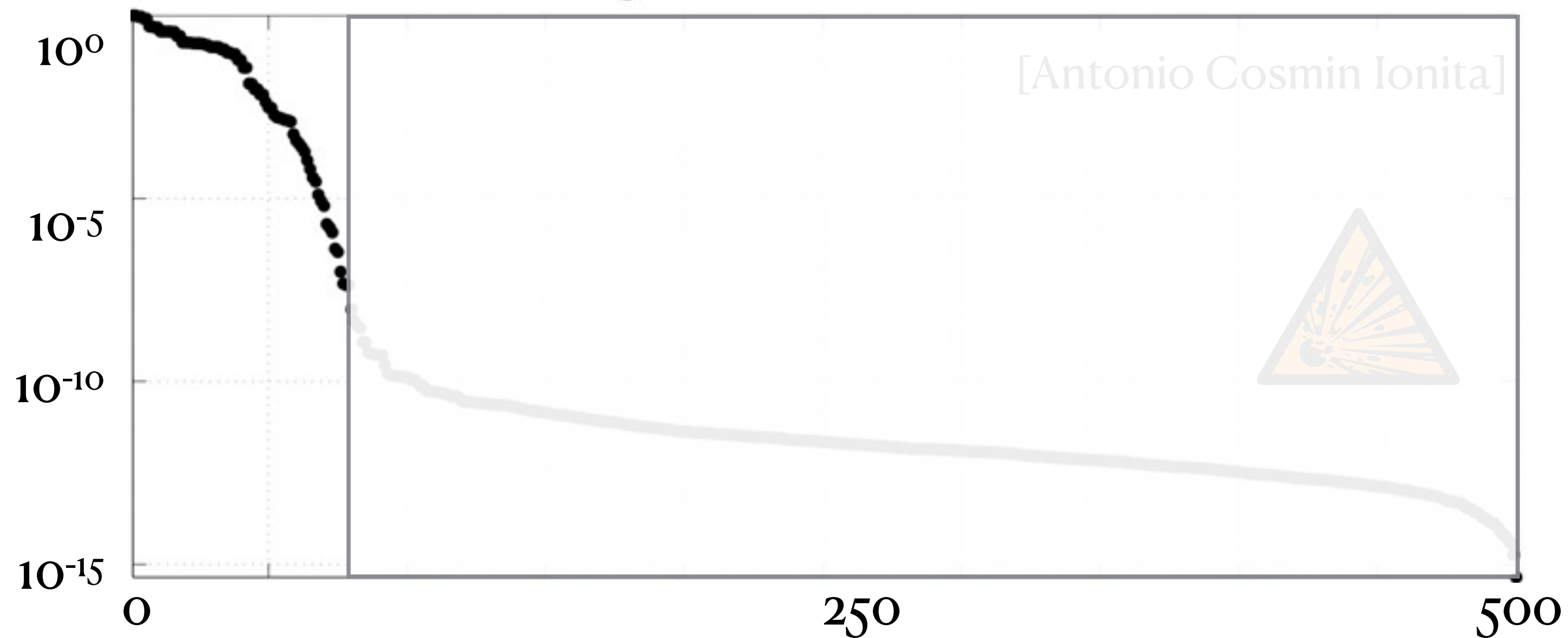
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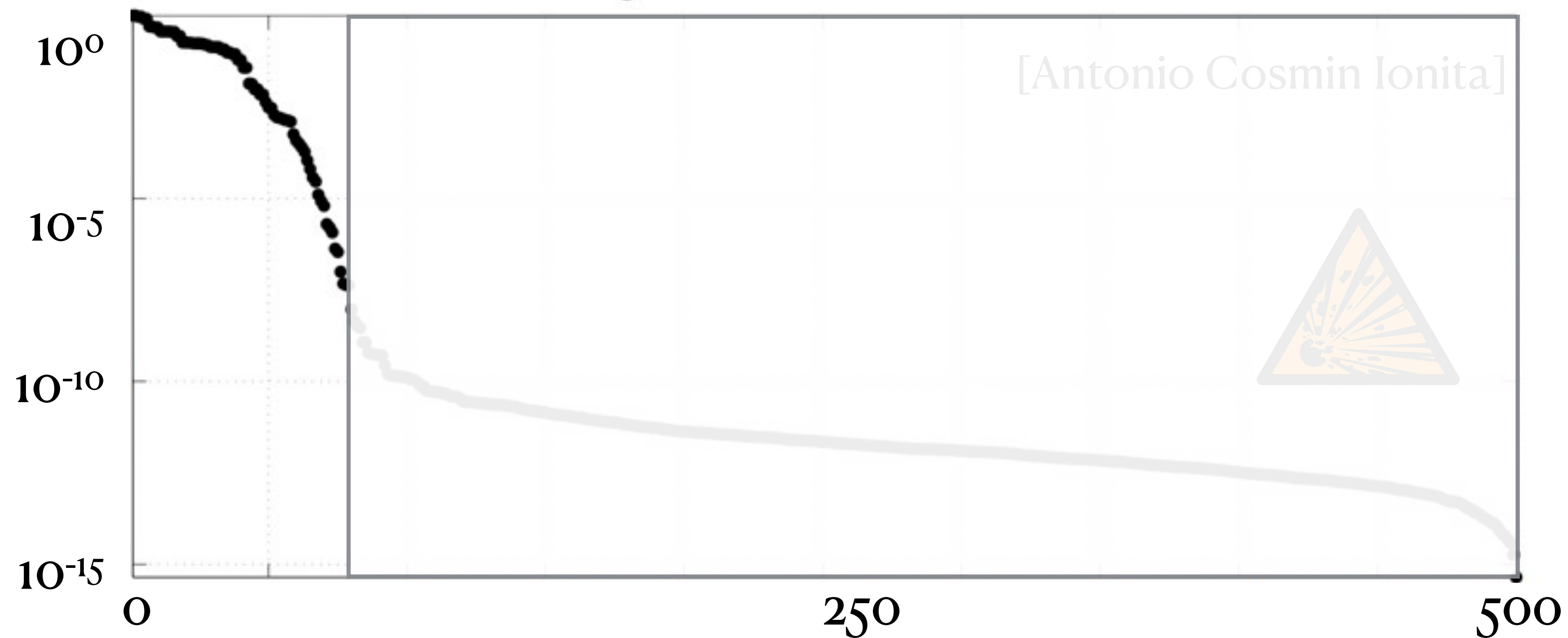
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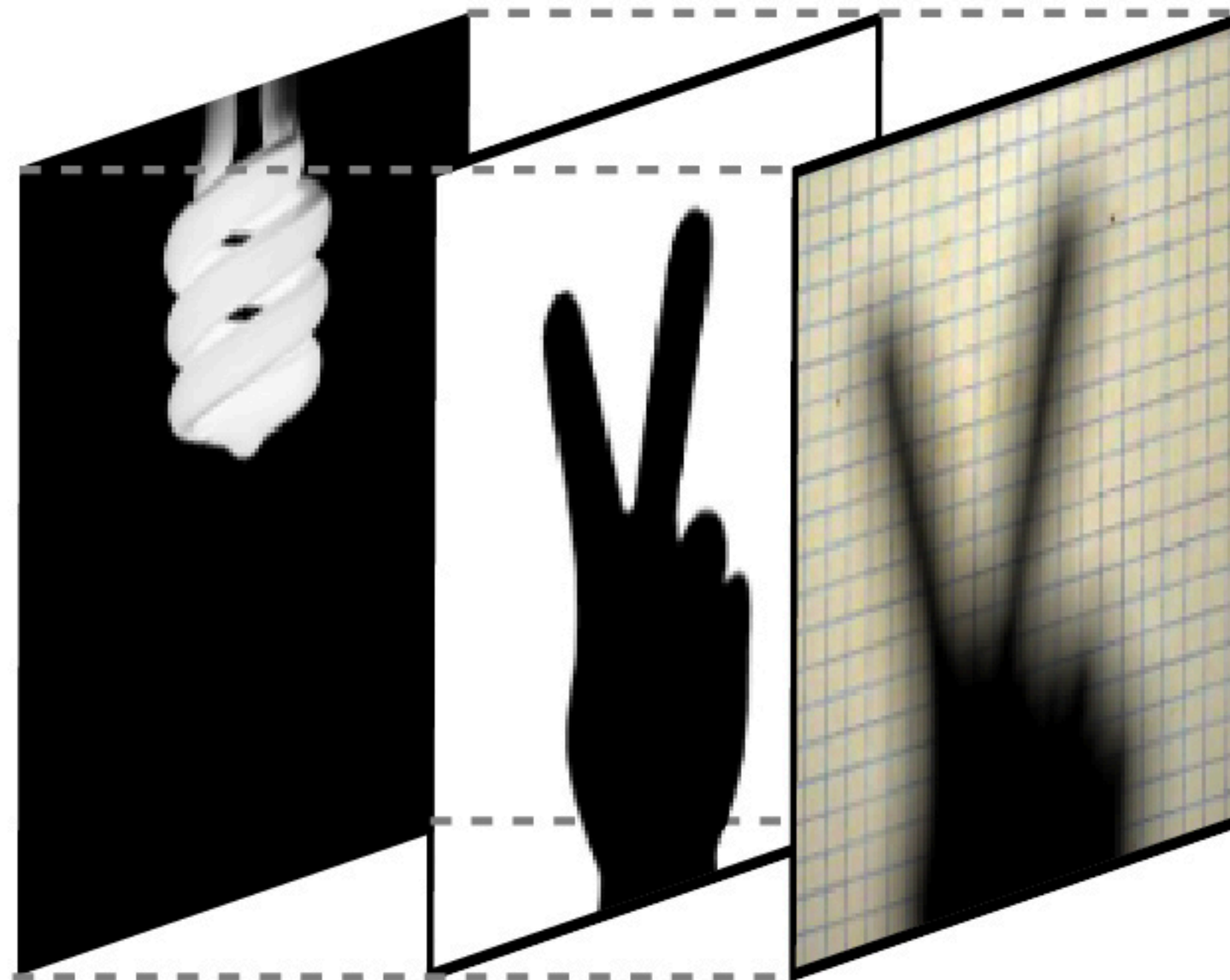
Computing inverses via the SVD

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T = \sum_{i=1}^n \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T \approx \sum_{i=1}^{n_{\max}} \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$$

Plot of singular values (in decreasing magnitude)

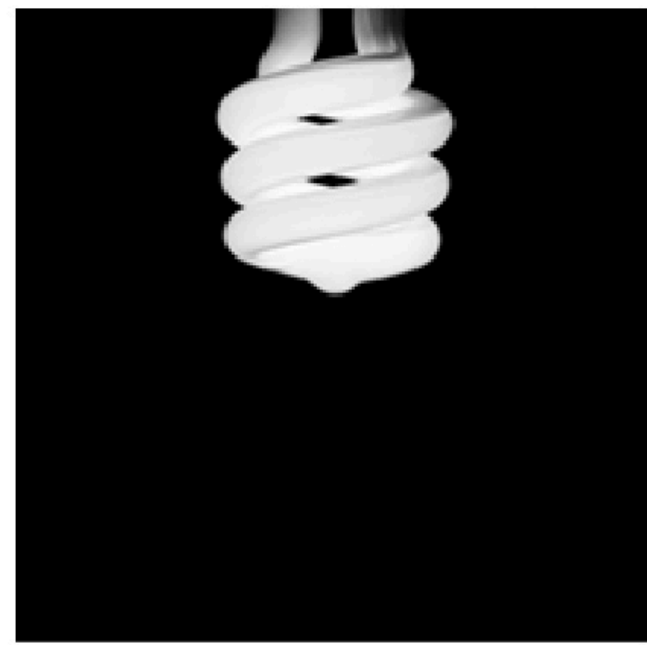


SVD as a regularization strategy



The SVD-powered X-ray glasses
(originally by Doug James @ Stanford U.)

fluorescent



Unknown



Unknown



Unknown

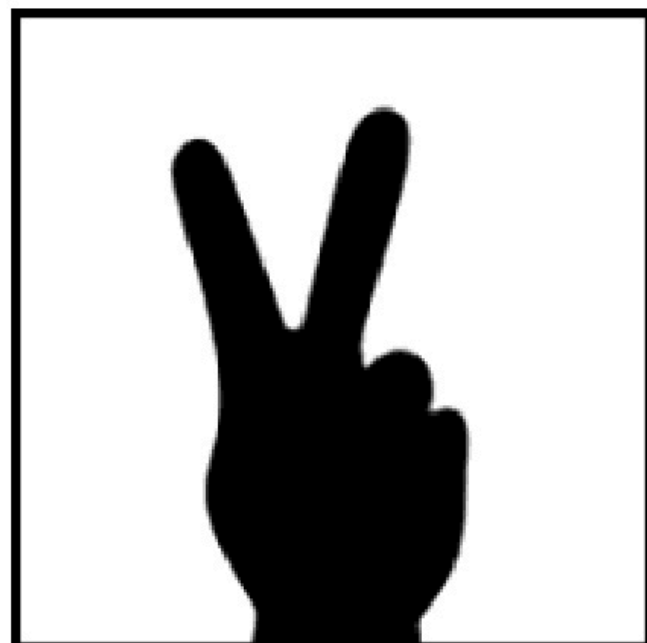


Unknown

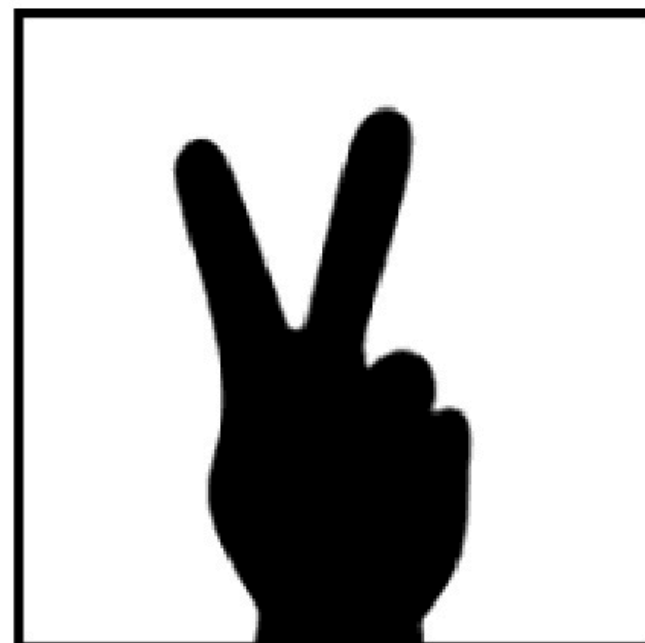


Light sources

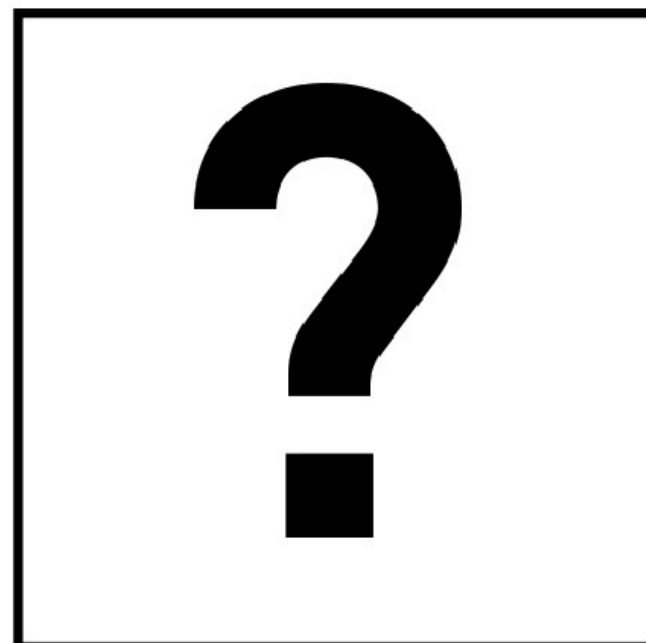
hand



hand



qmark



decoupage

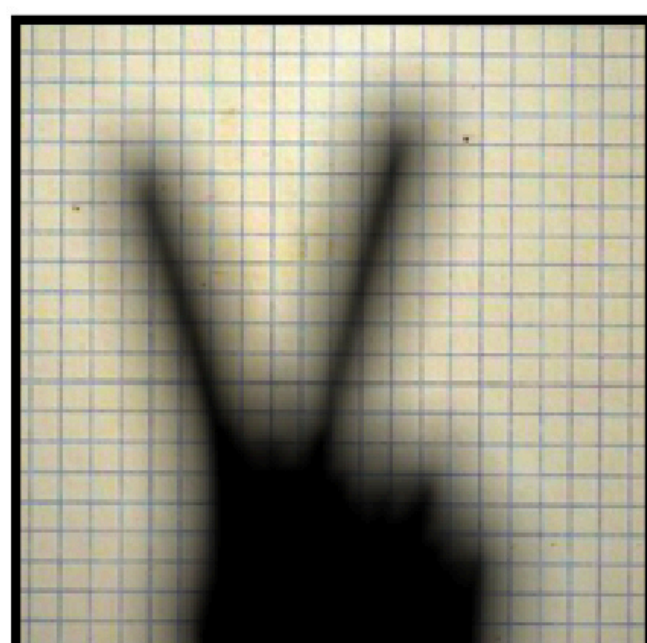


epfl

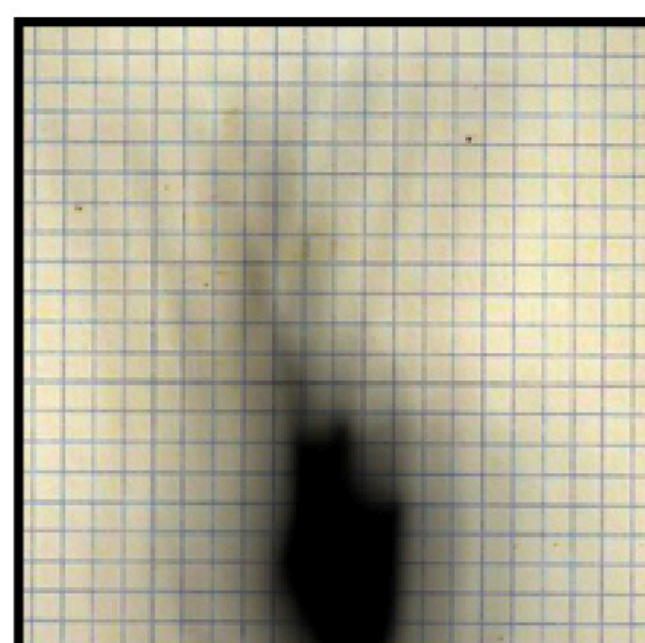


Blockers

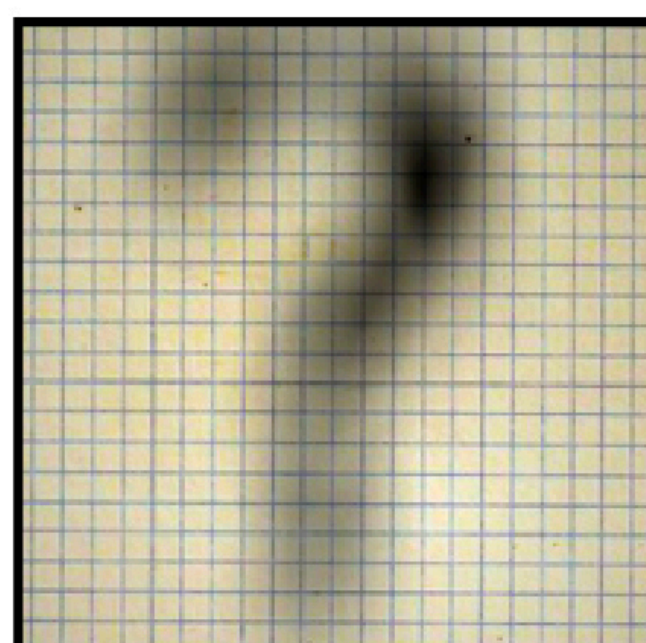
shadow1



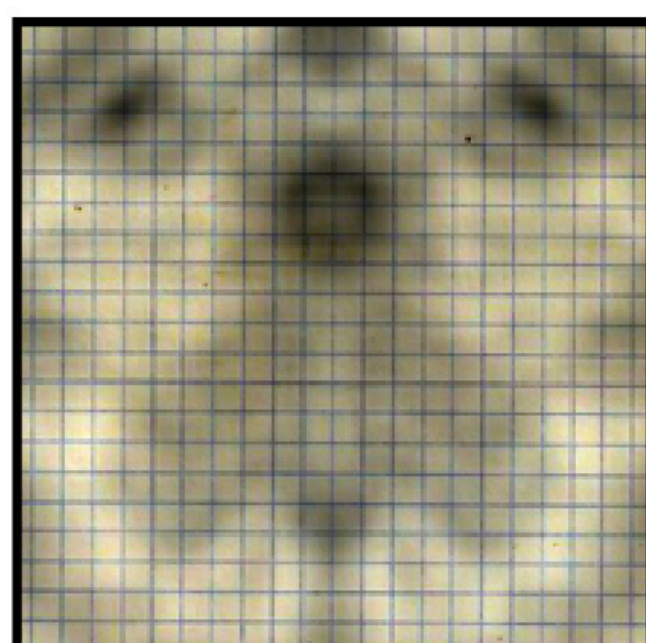
shadow2



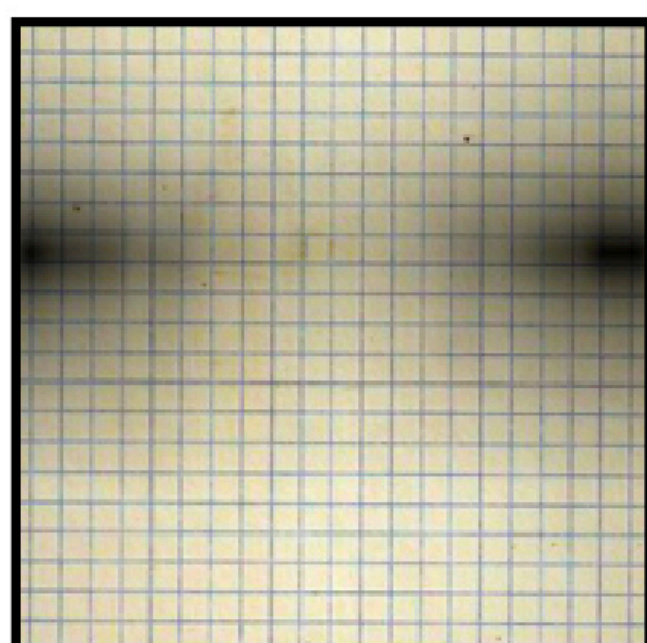
shadow3



shadow4

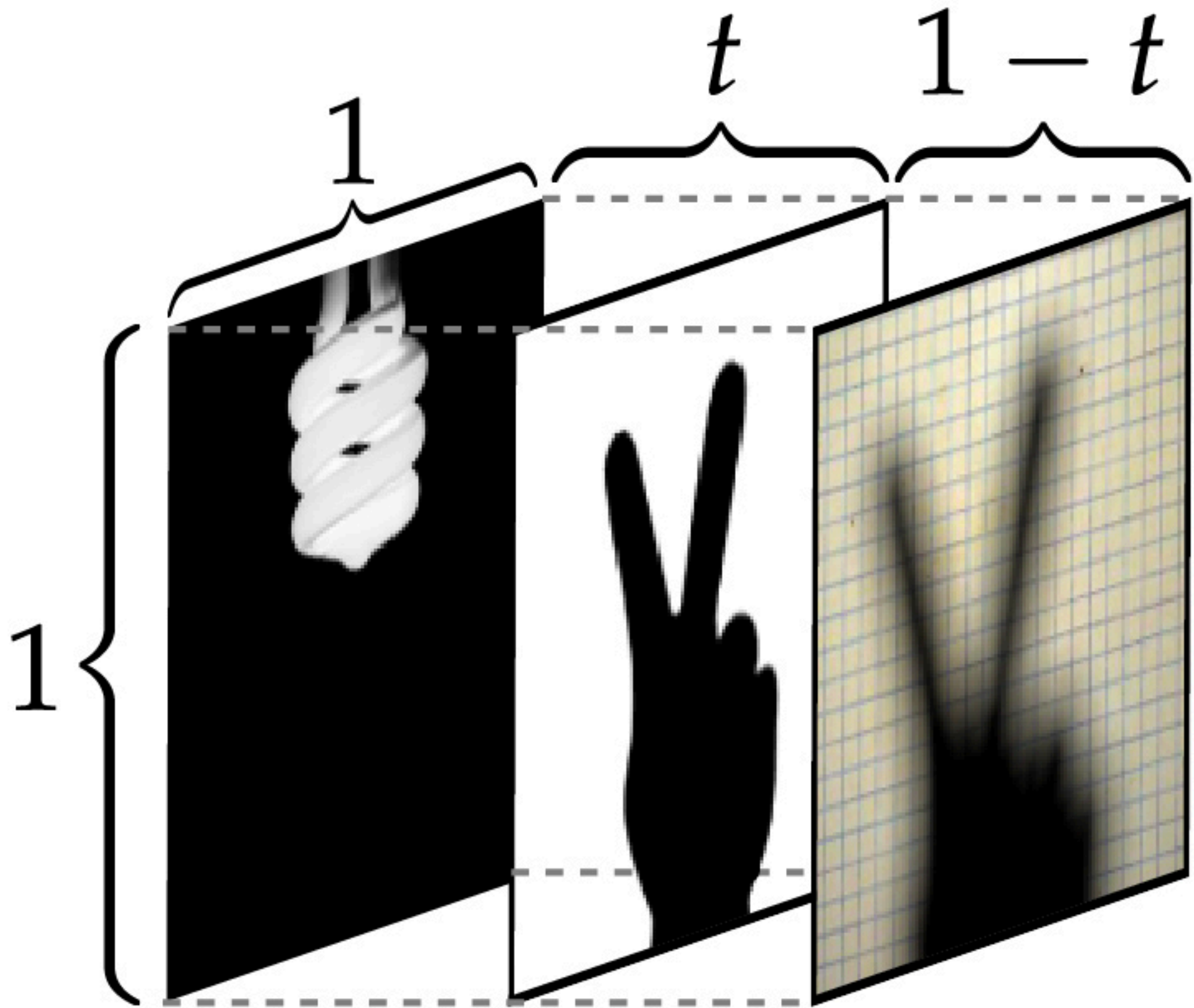


shadow5

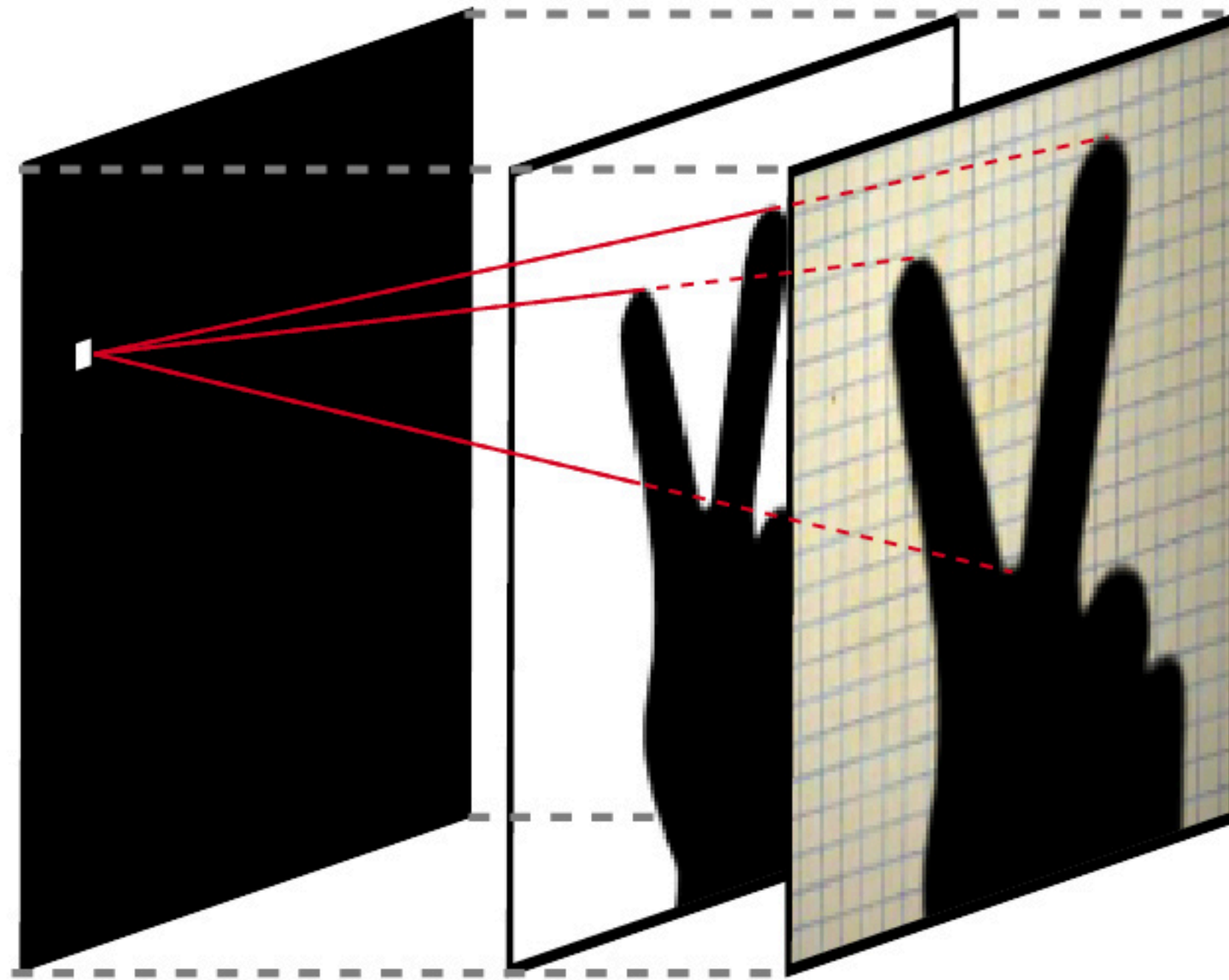
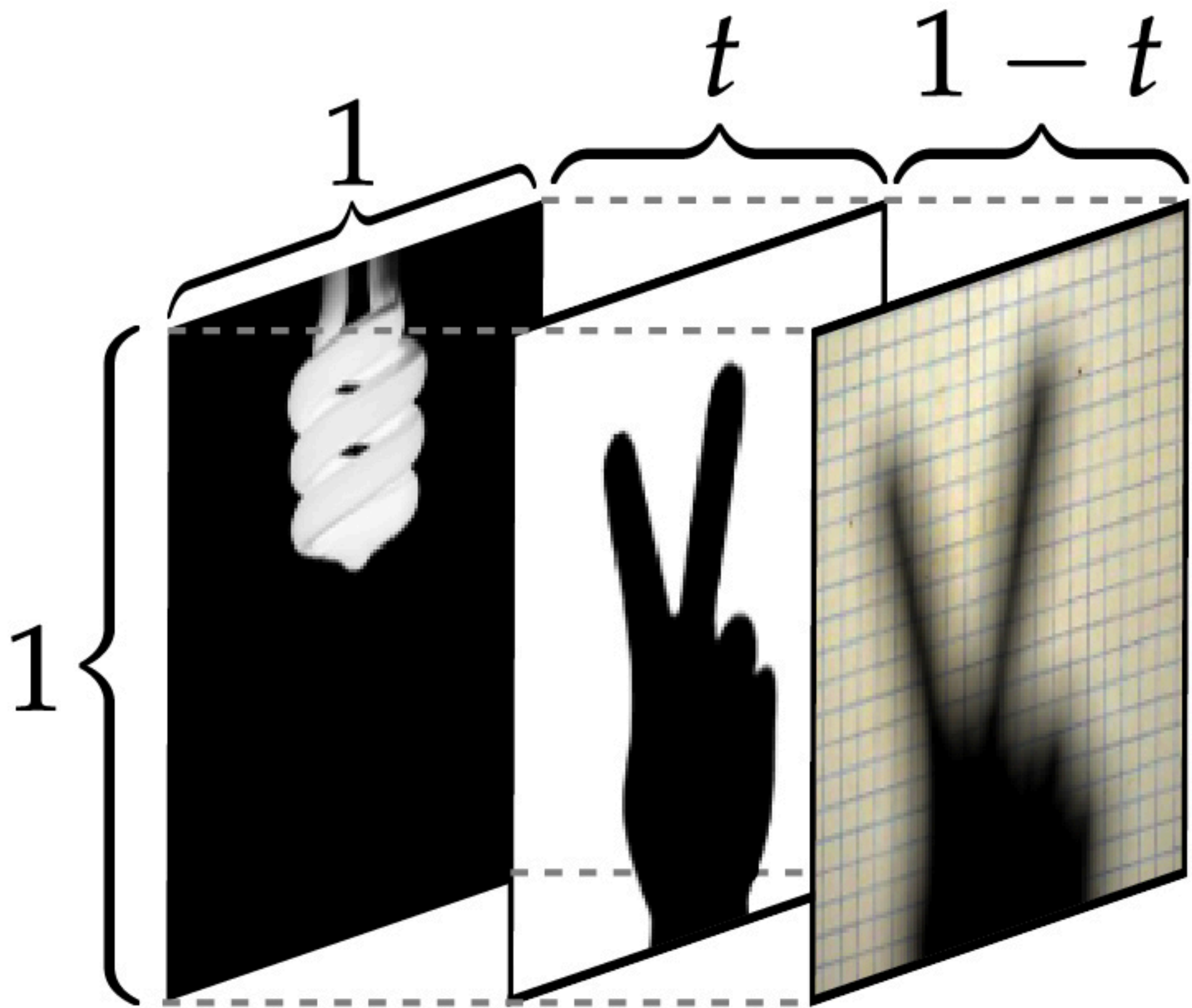


Shadow

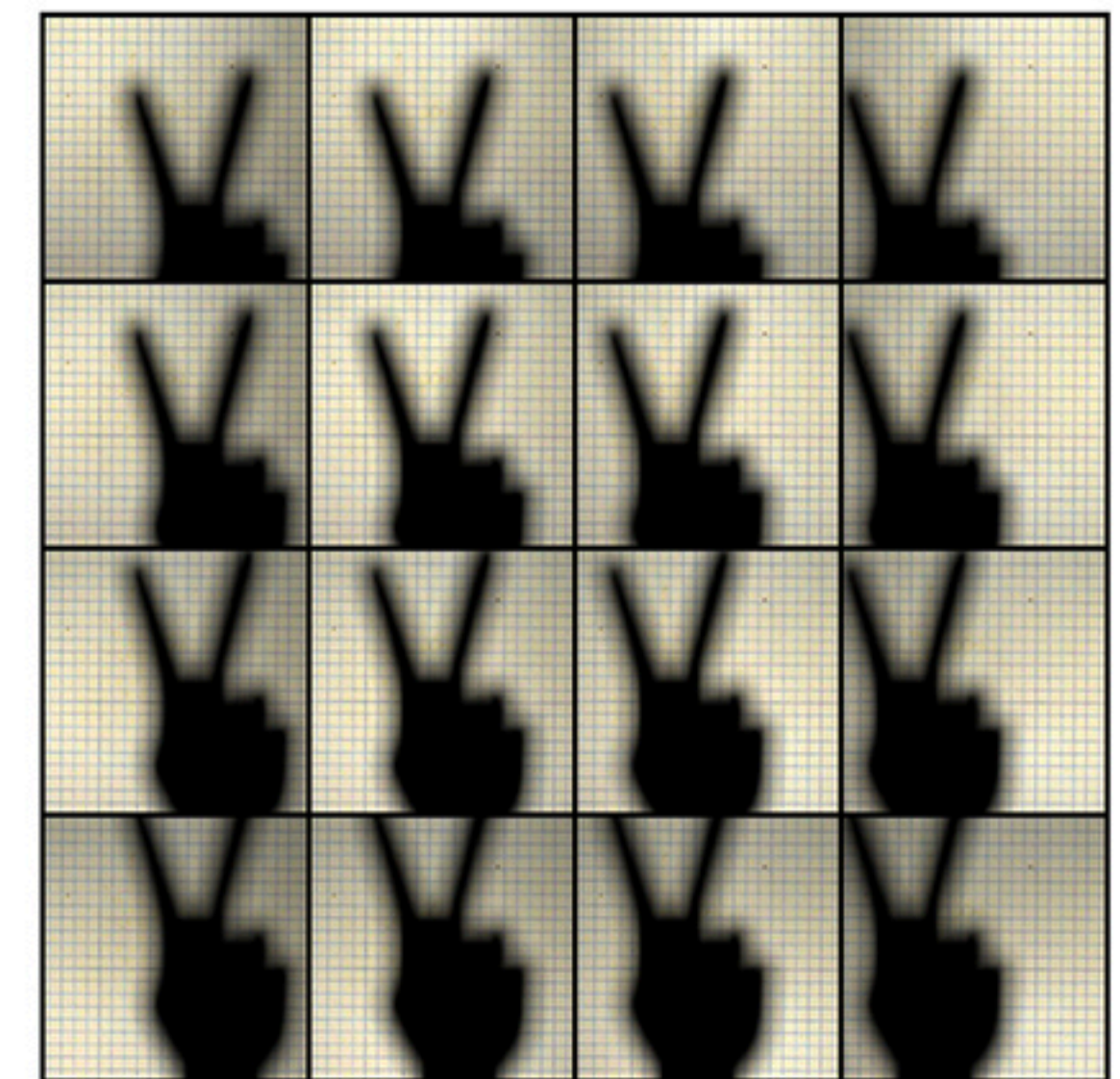
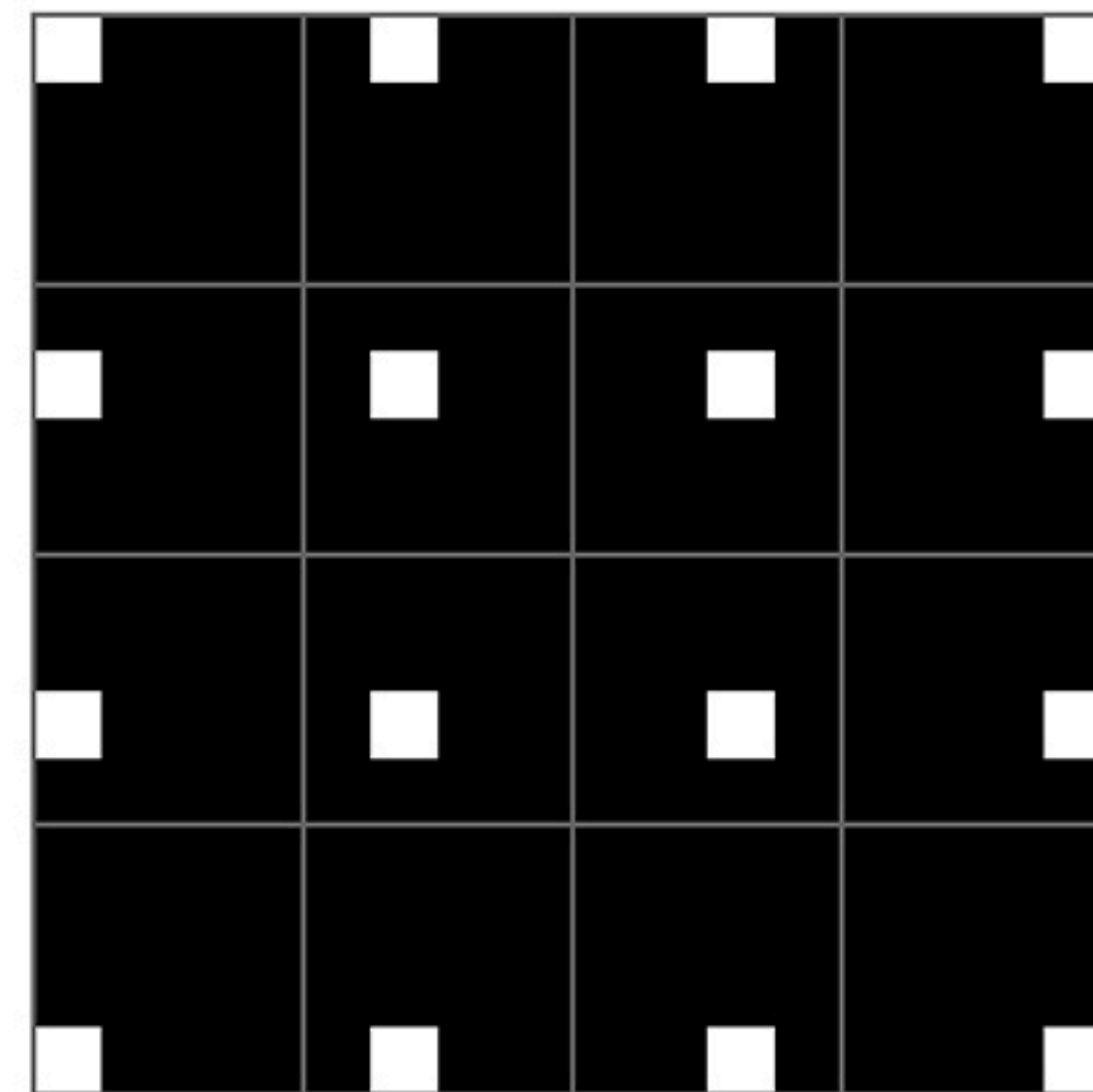
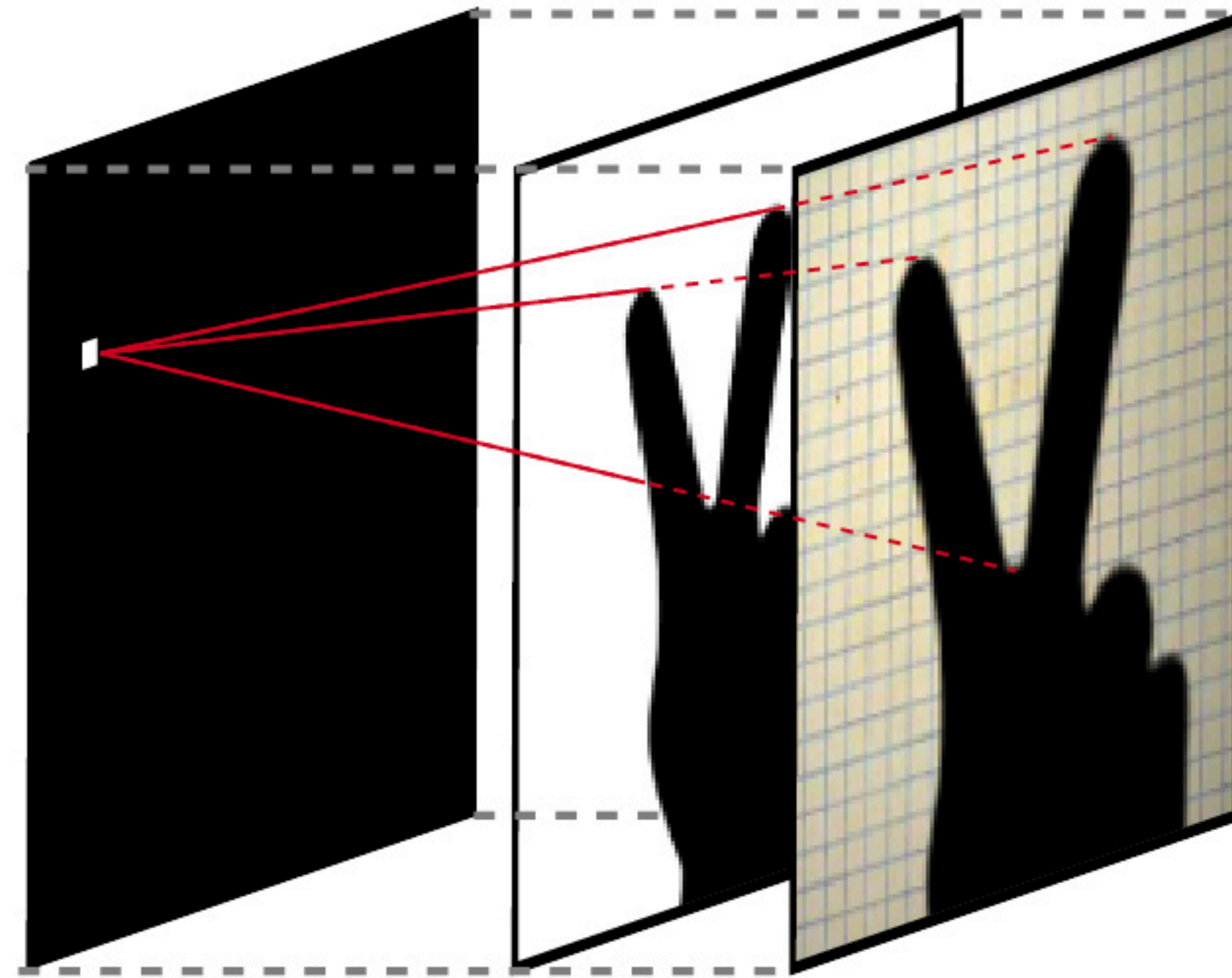
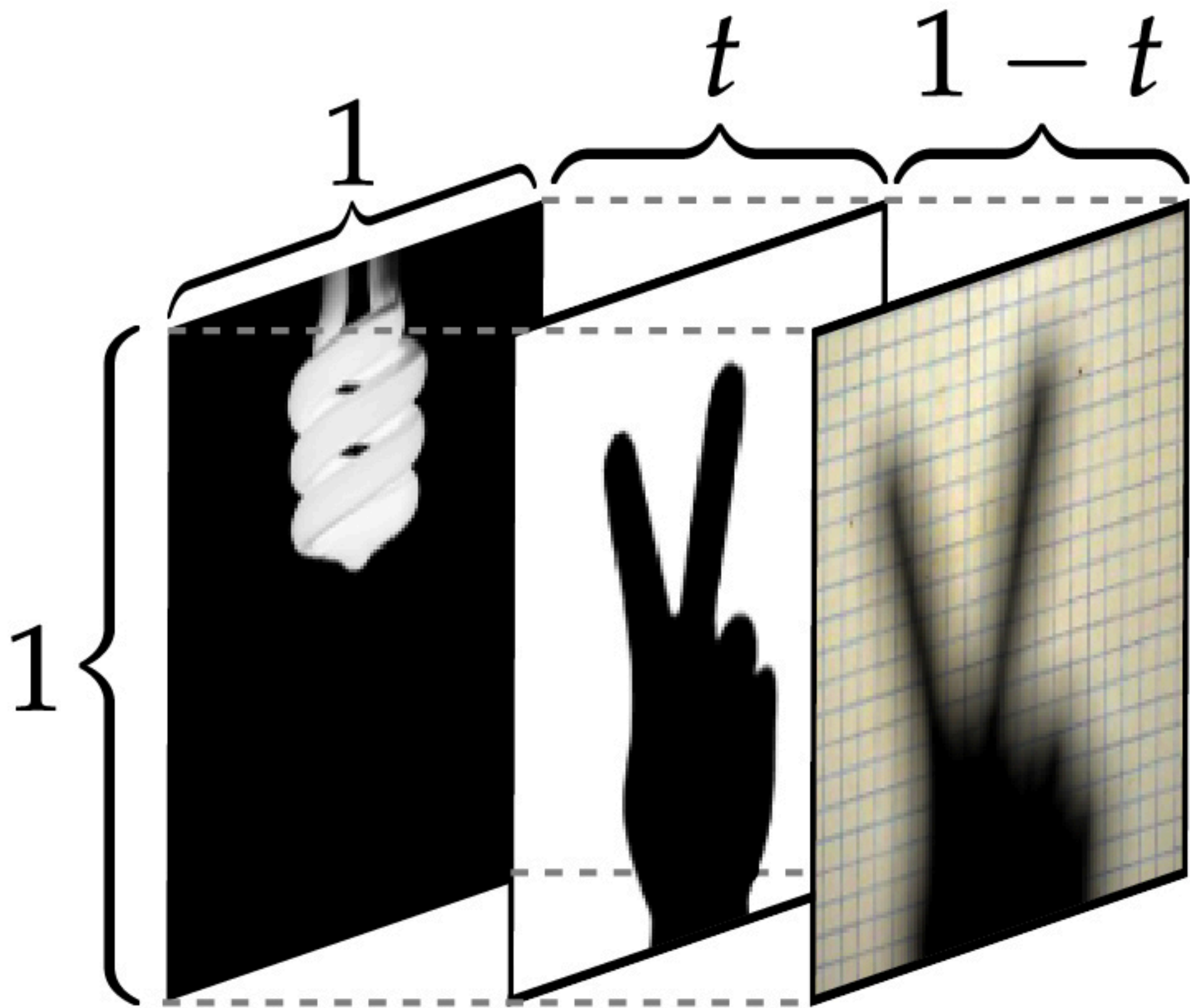
Problem setup



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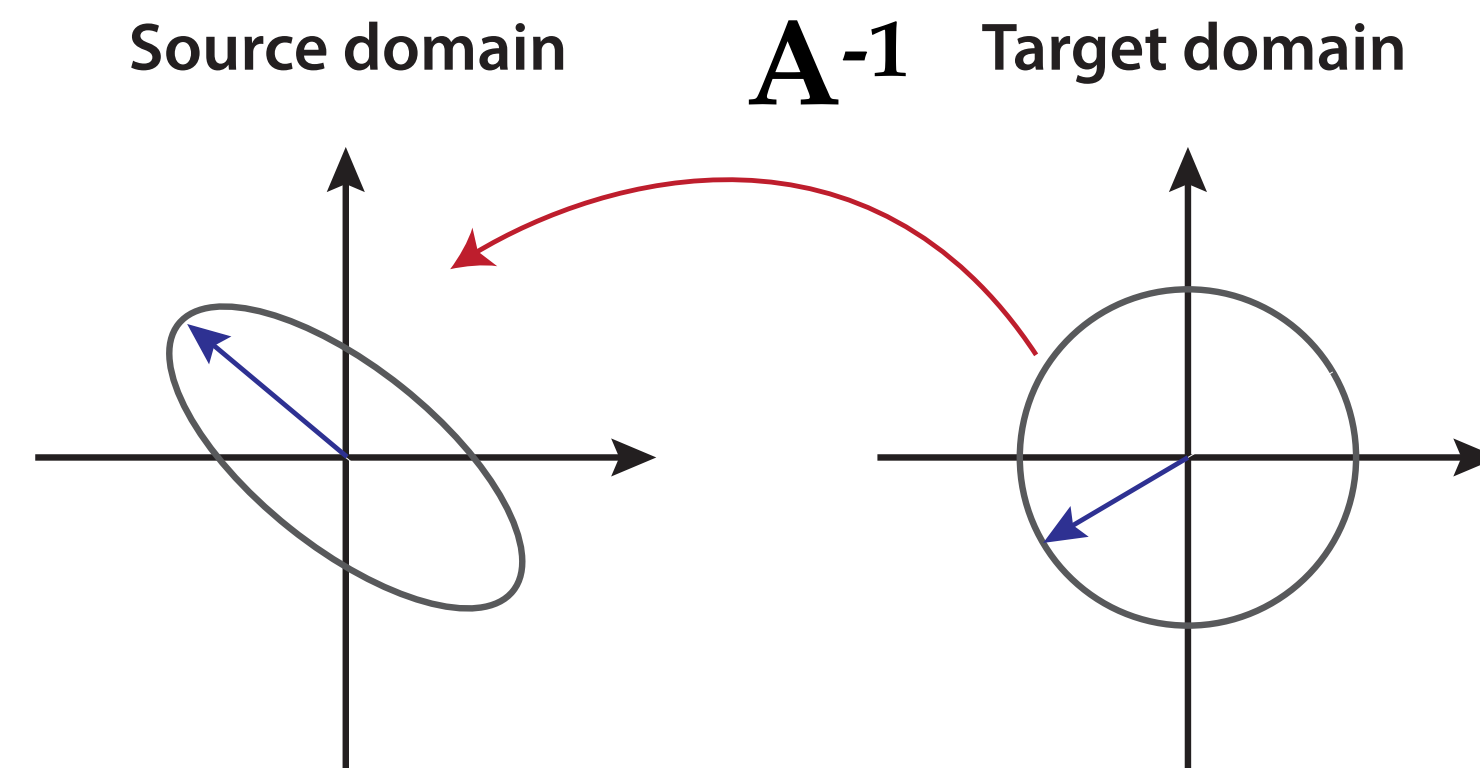
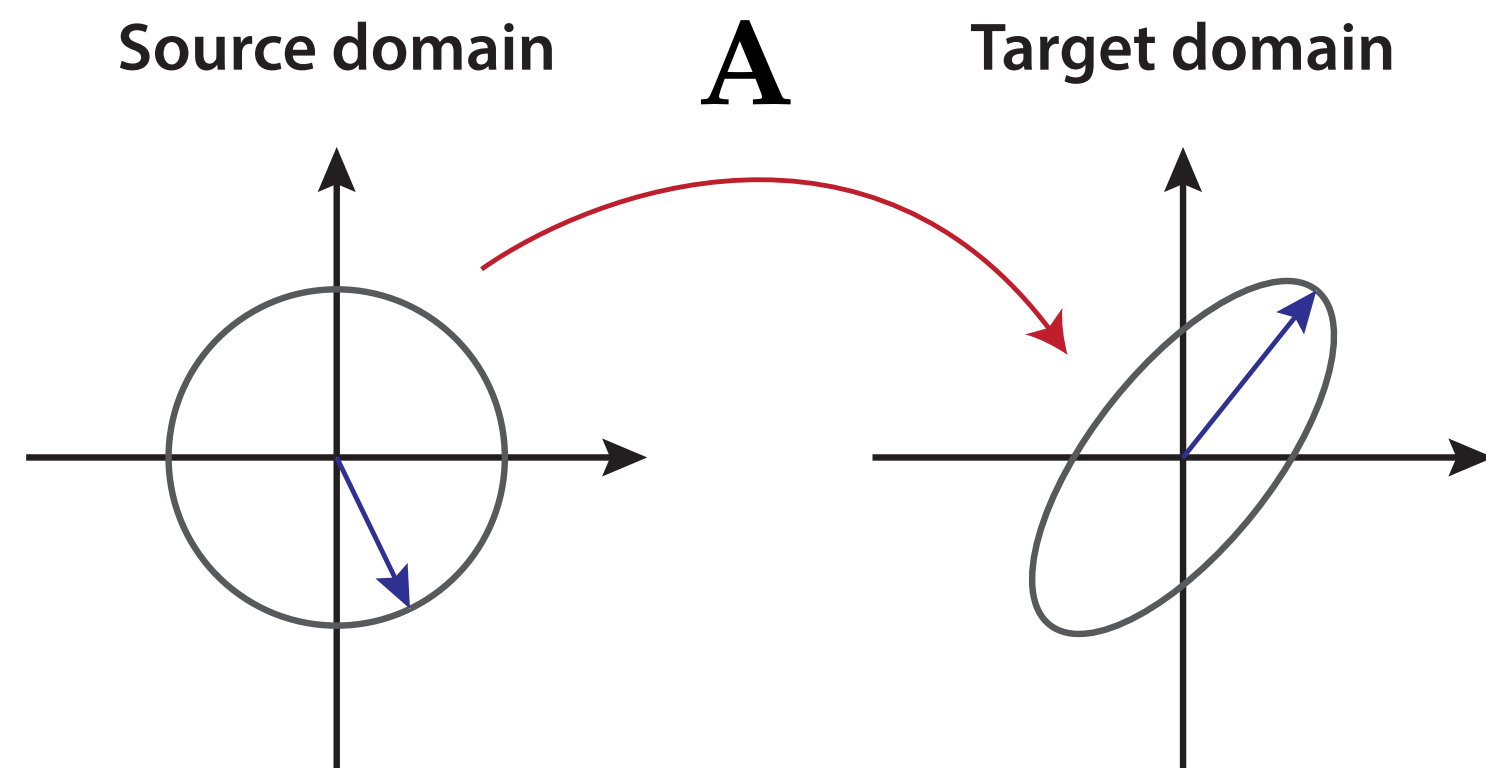
Problem setup



Demo time

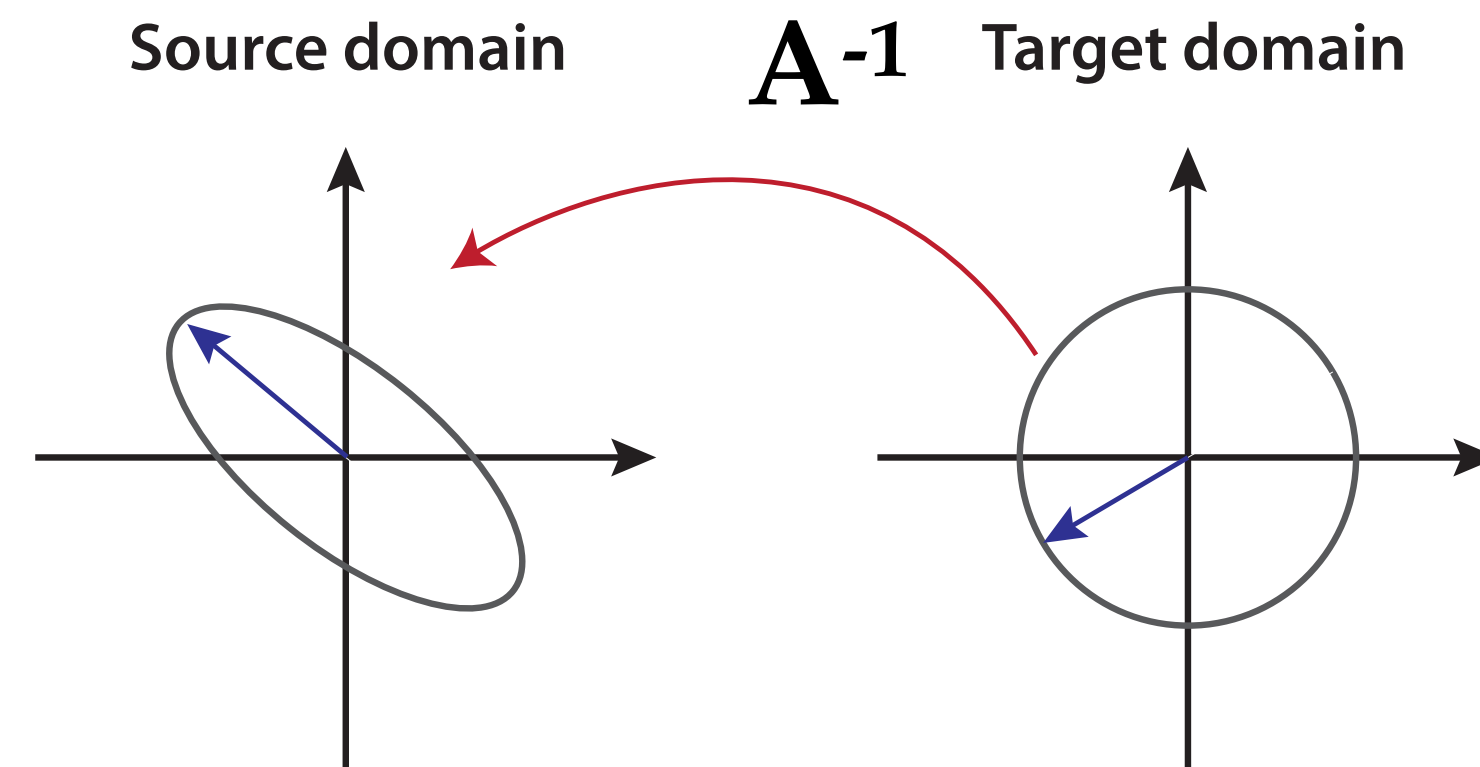
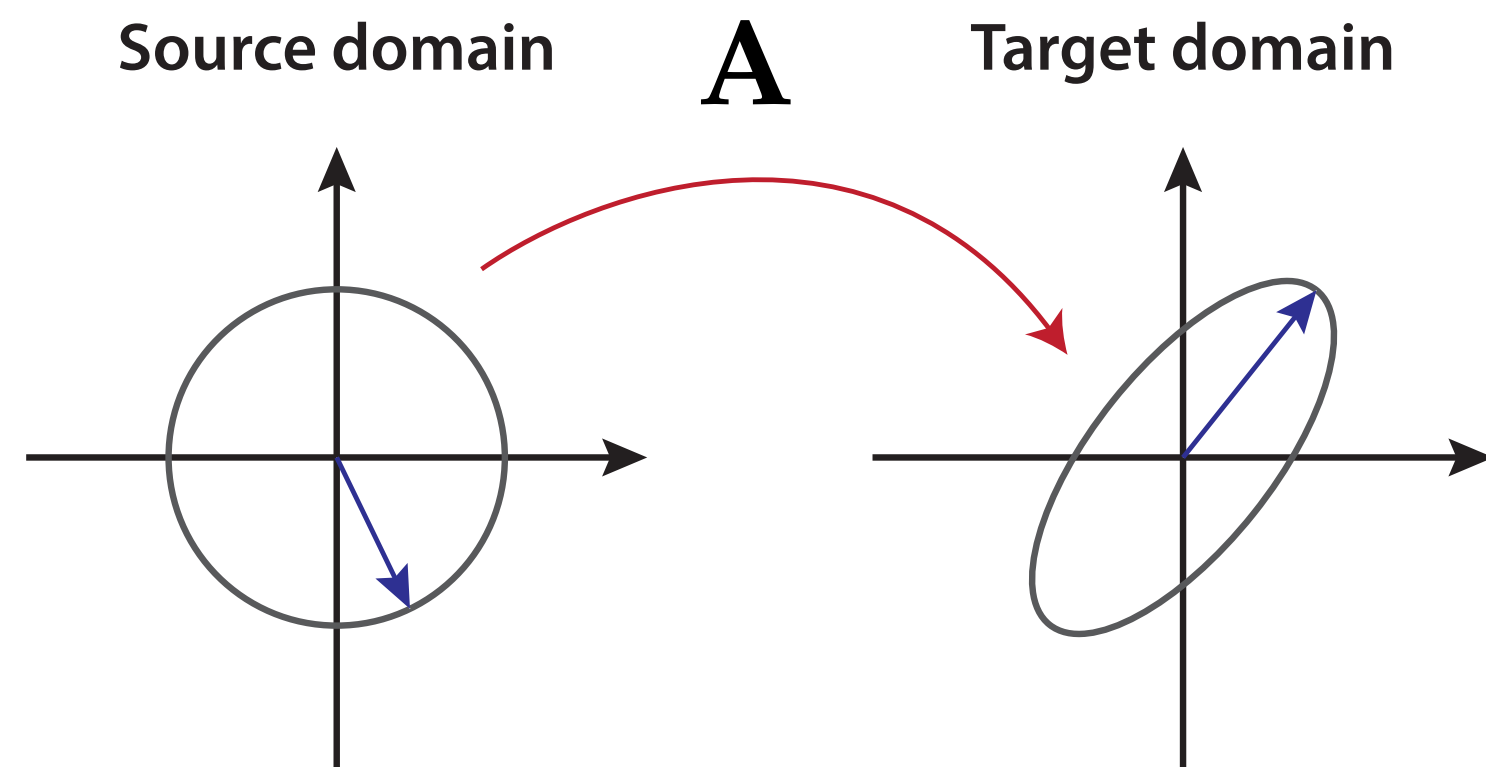
Revisiting the condition number

$$\text{cond}(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$$



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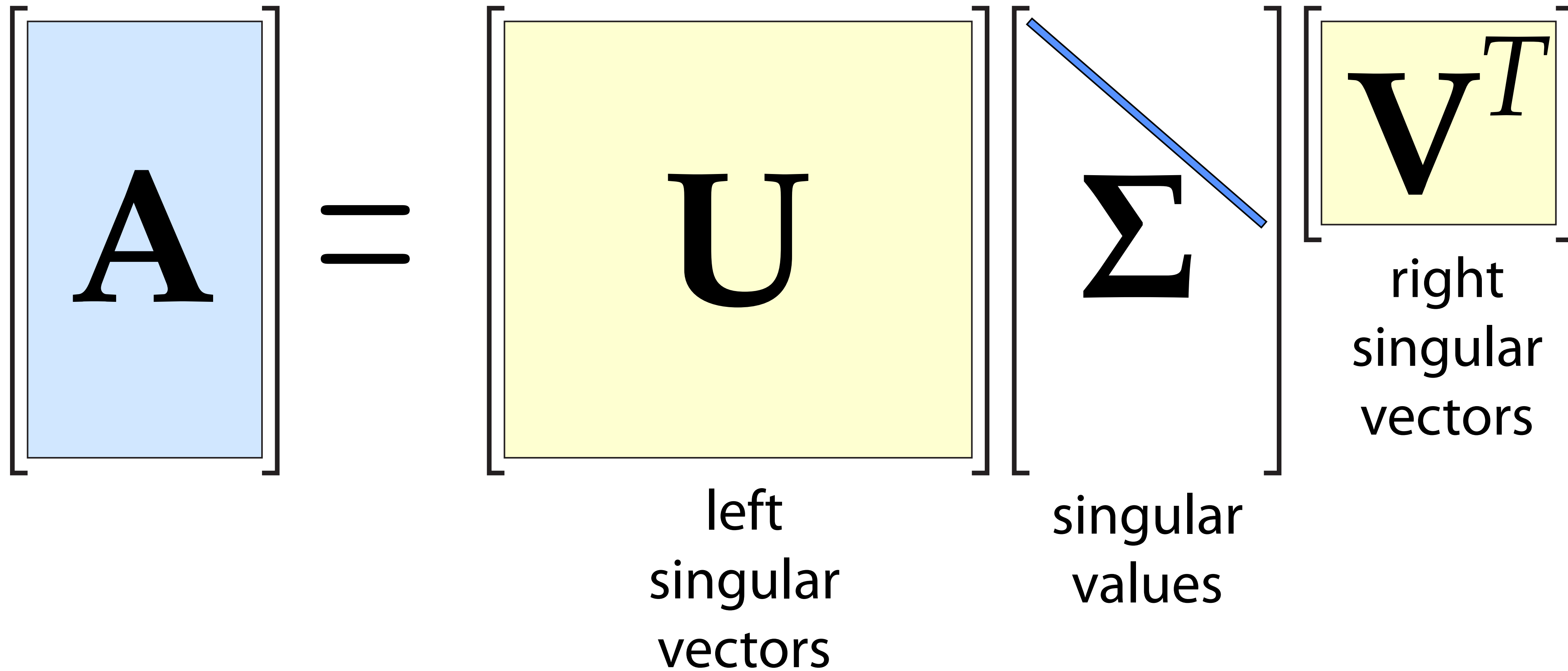
$$\begin{aligned}\text{cond}(\mathbf{A}) &= \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\| \\ &= \frac{\sigma_1}{\sigma_n}.\end{aligned}$$



SVD shape (tall & wide case)

$$\begin{bmatrix} \mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{V}^T \end{bmatrix}$$

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The Pseudoinverse

SVD provides the "ultimate" form of a matrix inverse

$$\mathbf{A}^+ = \sum_{i=1}^{\min\{n,m\}} \mathbf{v}_i \mathbf{u}_i^T \begin{cases} \frac{1}{\sigma_i}, & \sigma_i \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

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1. Identical to inverse for square & full-rank matrices.
2. Gives least-squares solution for overconstrained / tall linear systems
3. Gives minimum-norm solution for underconstrained / wide linear systems

For homework: Newton's method N dimensions

(Will go in more detail in next week's lecture)

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1D case:

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1D case:

N-D case:

For homework: Newton's method N dimensions

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1D case:

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

N-D case:

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N-D case:

$$\mathbf{x}_k = \mathbf{x}_{k-1} - [\nabla \mathbf{f}(\mathbf{x}_{k-1})]^{-1} \mathbf{f}(\mathbf{x}_{k-1})$$